TECHNICAL SCIENCE
RESEARCH IN UZBEKISTAN
ISSN (E): 2992-9148 SJIF 2024 = 5.333
ResearchBib Impact Factor: 9.576 / 2024
VOLUME-2, ISSUE-7

GAMILTON – KELI TEOREMASI

M.B.OTAMURODOV
O‘zbekiston-Finlandiya pedagogika instituti 2-kurs talabasi
mehrojiddinotamurodov@gmail.com
S.P.PO‘LATOV
O‘zbekiston-Finlandiya pedagogika instituti ‘‘Matematika-informatika” kafedrasi assistenti
suratjonpolatov@gmail.com

Annotatsiya: Ushbu maqola Gamilton – Keli teoremasining isbotlariga bag‘ishlangan bo‘lib, bunda teorema orqali bir nechta olimpiada misollari ishlab tushuntirilgan. Bundan tashqari Matritsaning xarakteristik ko‘phadi va Schur uchburchak teoremasi ham keltirib o‘tilgan.

Kalit so‘zlar: Matritsa, Unitar matritsa, birlik matritsa, nol matritsa, transponir, xos son, matritsa izi, matritsa determinanti, matritsaning xarakteristik ko‘phadi, yuqori uchburchakli matritsa.

Ushbu maqolada Unitar matritsa, Schur uchburchak teoremasi va Matritsaning xarakteristik ko‘phadi haqida aytib o‘tilgan. Matritsaning xarakteristik ko‘phadini tuzishga oid misollar ishlab ko‘rsatilgan. Shu bilan birga talabalar o‘rtasidagi olimpiadalarda juda keng qo‘llaniladigan Gamilton – Keli teoremasi isboti va ba’zi olimpiada misollari ishlab ko‘rsatilgan.

Ta’rif. $U \in M_n(\mathbb{C})$ matritsa berilgan bo‘lsin. U^* orqali U ning qo‘shma transponirini (U ni transponirlab har bir elementining qo‘shmasini olganimizni, ravshanki haqiqiy matritsalarda $U^* = U^T$) belgilaymiz. Agar $UU^* = I$ bo‘lsa U ga *Unitar matritsa* deyiladi. Bunda I U ning o‘lchamiga mos *birlik matritsa*.

Teorema. (Schur uchburchak teoremasi) Har bir $A \in M_n(\mathbb{C})$ matritsani $A = UTU^*$ shaklga keltirish mumkin, bunda U unitar matritsa, T esa yuqori uchburchakli matritsa (diagonalidan pastki elementlari 0 ga teng bo‘lgan matritsa).

Ta’rif. Ushbu $P(x) = \det(xI_n - A)$ ko‘phadga $A \in M_n(\mathbb{C})$ *Matritsaning xarakteristik ko‘phadi* deyiladi. Ravshanki xarakteristik ko‘phadning ildizlari A matritsaning $\lambda_1, \lambda_2, ..., \lambda_n$ xos sonlaridan iborat. Shunga ko‘ra $P(x)$ bu yerda x ni kompleks son shaklda ham yozsak bo‘ladi. Ko‘rinib turibdiki $P(x)$ ko‘phad ushbu shaklda $\mathbb{C}$ to‘plamni $\mathbb{C}$ to‘plamga akslantiradi. Agar bu yerdagi x o‘rniga son emas matritsa qo‘ysak ravshanki endi $P(x)$ ko‘phad $M_n(\mathbb{C})$ to‘plamni $M_n(\mathbb{C})$ ga akslantiradi va uning shakli ham $P(X) = (X - \lambda_1 I)(X - \lambda_2 I)\cdots(X - \lambda_n I)$ bu yerda X matritsaga o‘zgaradi.

ISSN (E): 2992-9148 SJIF 2024 = 5.333
ResearchBib Impact Factor: 9.576 / 2024
VOLUME-2, ISSUE-7

$A \in M_2(R)$ matritsaning xarakteristik ko‘phadining umumiy ko‘rinishi:

$$|xI - A| = 0 \quad \text{yoki} \quad \lambda^2 - x_1\lambda + x_2 = 0.$$

Misol. Berilgan matritsaning xarakteristik ko‘phadini tuzing: $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$.

Yechim. $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ matritsa xarakteristik ko‘phadining umumiy ko‘rinishi quyidagicha: $\lambda^2 - x_1\lambda + x_2 = 0$. Bu yerda $x_1 = 1 + 2 = 3$ (bosh dioganal elementlari yig‘indisi), $x_2 = \det|A| = 1\cdot 2 - 0\cdot 2 = 2$ (A matritsaning determinanti). Demak xarakteristik ko‘phadning umumiy ko‘rinishi quyidagicha ekan: $\lambda^2 - 3\lambda + 2 = 0$.

$A \in M_3(R)$ matritsaning xarakteristik ko‘phadining umumiy ko‘rinishi:

$$|xI - A| = 0 \quad \text{yoki} \quad \lambda^3 - x_1\lambda^2 + x_2\lambda - x_3 = 0.$$

Misol. Berilgan matritsaning xarakteristik ko‘phadini tuzing:

$$A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}.$$

Yechim. $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ matritsaning xarakteristik ko‘phadining umumiy ko‘rinishi quyidagicha: $\lambda^3 - x_1\lambda^2 + x_2\lambda - x_3 = 0$. Bu yerda $x_1 = 8 + 7 + 3 = 18$ (bosh dioganal elementlari yig‘indisi), $x_2 = \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} + \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} = 5 + 20 + 20 = 45$ (bosh dioganal elementlari kombinatsiyasidagi 2×2 matritsalar determinantlari yig‘indisi), $x_3 = \det|A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} = -60 + 40 + 20 = 0$ (A matritsaning determinanti).

Demak xarakteristik ko‘phadning umumiy ko‘rinishi quyidagicha ekan: $\lambda^3 - 18\lambda^2 + 45\lambda = 0$.

Misol. Berilgan matritsaning xarakteristik ko‘phadini tuzing: $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ va $P(A) = O$ ni tekshiring.

Yechim. $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ matritsaning xarakteristik ko‘phadini $|xI - A| = 0$ bu ko‘rinishda qidirsak.

ISSN (E): 2992-9148 SJIF 2024 = 5.333
ResearchBib Impact Factor: 9.576 / 2024
VOLUME-2, ISSUE-7

$$|xI-A|=\left|x\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}1&2\\3&4\end{pmatrix}\right|=\begin{vmatrix}x-1&-2\\-3&x-4\end{vmatrix}=(x-1)\cdot(x-4)-6=x^2-5x-2=0.$$

Endi $P(A)=O$ mi tekshirsak:

$$P(A)=A^2-5A-2I=\begin{pmatrix}1&3\\2&4\end{pmatrix}\begin{pmatrix}1&3\\2&4\end{pmatrix}-5\begin{pmatrix}1&3\\2&4\end{pmatrix}-2\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}7&15\\10&22\end{pmatrix}-\begin{pmatrix}5&15\\10&20\end{pmatrix}-\begin{pmatrix}2&0\\0&2\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Teorema. (Gamilton – Keli teoremasi) $P(x)$ A matritsaning xarakteristik ko‘phadi bo‘lsa $P(A)=O$ tenglik o‘rinli. Bu yerda O A ning o‘lchamiga mos *nol matritsa*.

Isbot. Ta’rifga ko‘ra $P(A)=(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I)$. Biz o‘ng tomondagi matritsalar ko‘paytmasini 0 ga tengligini ko‘rsatishimiz kerak. Schur uchburchak teoremasiga ko‘ra biz A ni $A=UTU^*$ shaklga keltira olamiz (U unitar, T yuqori uchburchakli matritsalar). Bundan foydalansak

$$P(A)=(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I)=$$
$$=\left(UTU^*-\lambda_1 I\right)\left(UTU^*-\lambda_2 I\right)\cdots\left(UTU^*-\lambda_n I\right)=$$
$$=\left(UTU^*-\lambda_1 UU^*\right)\left(UTU^*-\lambda_2 UU^*\right)\cdots\left(UTU^*-\lambda_n UU^*\right)=$$
$$=U(T-\lambda_1 I)U^*U(T-\lambda_2 I)U^*U(T-\lambda_3 I)U^*\cdots U(T-\lambda_n I)U^*=$$
$$=U(T-\lambda_1 I)(T-\lambda_2 I)(T-\lambda_3 I)\cdots(T-\lambda_n I)U^*=$$

[T ning yuqori uchburchakli matritsa ekanligini hisobga olsak, bilamizki T ning dioganalida $\lambda_1,\lambda_2,...,\lambda_n$ xos sonlar joylashadi.]

ISSN (E): 2992-9148 SJIF 2024 = 5.333
ResearchBib Impact Factor: 9.576 / 2024
VOLUME-2, ISSUE-7

$$=U\begin{pmatrix}0&*&*&\cdots&*\\0&\lambda_2&*&\cdots&*\\0&0&\lambda_3&\cdots&*\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&\lambda_n\end{pmatrix}\begin{pmatrix}\lambda_1&*&*&\cdots&*\\0&0&*&\cdots&*\\0&0&\lambda_3&\cdots&*\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&\lambda_n\end{pmatrix}\begin{pmatrix}\lambda_1&*&*&\cdots&*\\0&\lambda_2&*&\cdots&*\\0&0&0&\cdots&*\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&\lambda_n\end{pmatrix}\cdots\begin{pmatrix}\lambda_1&*&*&\cdots&*\\0&\lambda_2&*&\cdots&*\\0&0&\lambda_3&\cdots&*\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&0\end{pmatrix}U^*=$$

$$=U\begin{pmatrix}0&0&*&\cdots&*\\0&0&*&\cdots&*\\0&0&\lambda_3&\cdots&*\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&\lambda_n\end{pmatrix}\begin{pmatrix}\lambda_1&*&*&\cdots&*\\0&\lambda_2&*&\cdots&*\\0&0&0&\cdots&*\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&\lambda_n\end{pmatrix}\cdots\begin{pmatrix}\lambda_1&*&*&\cdots&*\\0&\lambda_2&*&\cdots&*\\0&0&\lambda_3&\cdots&*\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&0\end{pmatrix}U^*=\cdots=$$

$$=U\begin{pmatrix}0&0&0&\cdots&*\\0&0&0&\cdots&*\\0&0&0&\cdots&*\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&\lambda_n\end{pmatrix}\begin{pmatrix}\lambda_1&*&*&\cdots&*\\0&\lambda_2&*&\cdots&*\\0&0&\lambda_3&\cdots&*\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&0\end{pmatrix}U^*=U\begin{pmatrix}0&0&0&\cdots&0\\0&0&0&\cdots&0\\0&0&0&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&0\end{pmatrix}U^*=\begin{pmatrix}0&0&0&\cdots&0\\0&0&0&\cdots&0\\0&0&0&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&0\end{pmatrix}$$

Misol. A va B 2×2 matritsalarning determinanti 1 ga teng bo'lsin. U holda quyidagini isbotlang:

$$tr(AB)-tr(A)tr(B)+tr\left(AB^{-1}\right)=0.$$

Yechim. Gamilton – Keli teoremasiga ko'ra

$$B^2-tr(B)B+I_2=O_2.$$

Chap tomondan AB^{-1} ga ko'paytirsak:

$$AB-tr(B)A+AB^{-1}=O_2$$

Natijani olish uchun ikkala tomonni ham izini olib yuborsak.

$$tr(AB)-tr(A)tr(B)+tr\left(AB^{-1}\right)=0.$$

Misol. A va B 3×3 matritsalar bo'lsin. Isbotlang:

$$\det(AB-BA)=\frac{tr\left((AB-BA)^3\right)}{3}.$$

Yechim. Gamilton – Keli teoremasiga ko'ra

$$(AB-BA)^3-c_1(AB-BA)^2+c_2(AB-BA)-c_3I_3=O_3$$

Bu yerda $c_1=tr(AB-BA)=0$ va $c_3=\det(AB-BA)$. Izini olib, $AB-BA$ ni izi 0 ekanligidan foydalanib,

$$tr\left((AB-BA)^3\right)-3\det(AB-BA)=0$$

ISSN (E): 2992-9148 SJIF 2024 = 5.333
ResearchBib Impact Factor: 9.576 / 2024
VOLUME-2, ISSUE-7

ni olamiz va tenglik isbotlandi.

Misol. Har bir $A \in M_2(\mathbb{R})$ matritsa uchun $A = B^2 + C^2$ tenglik qanoatlantiradigan shunday $B, C \in M_2(\mathbb{R})$ matritsalar mavjudligini ko'rsating.

Yechim. 2×2 matritsalar Gamilton – Keli teoremasiga ko'ra quyidagi shartni qanoatlantiradi:

$$A^2 - (trA)A + (\det A)I = O_2 .$$

Cheksiz katta t lar uchun $\lim_{t\to+\infty} tr(A+tI) = +\infty$ va $\lim_{t\to+\infty} \frac{\det(A+tI)}{tr(A+tI)} - t = -\infty$ lar o'rinli.

$$A = (A+tI) - tI = \frac{1}{tr(A+tI)}(A+tI)^2 + \left(\frac{\det(A+tI)}{tr(A+tI)} - t\right)I =$$

$$= \left(\frac{1}{\sqrt{tr(A+tI)}}(A+tI)\right)^2 + \left(\sqrt{t - \frac{\det(A+tI)}{tr(A+tI)}}\right)^2(-I) =$$

$$= \left(\frac{1}{\sqrt{tr(A+tI)}}(A+tI)\right)^2 + \left(\sqrt{t - \frac{\det(A+tI)}{tr(A+tI)}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)^2 = B^2 + C^2$$

Demak $B = \frac{1}{\sqrt{tr(A+tI)}}(A+tI)$ va $C = \sqrt{t - \frac{\det(A+tI)}{tr(A+tI)}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ lar topilar ekan.

Foydalanilgan adabiyotlar:

1. www.mathresource.iitb.ac.in/linear%20algebra/chapter2.0.html
2. https://en.wikipedia.org/wiki/Matrix_(mathematics)
3. www.slideshare.net/moneebakhtar50/application-of-matrices-in-real-life
4. www.youtube.com/watch?v=jzHb1R5wWYU
5. www.clarkson.edu/~pmarzocc/AE430/Matlab_Eig.pdf