МЕДИЦИНА, ПЕДАГОГИКА И ТЕХНОЛОГИЯ:
ТЕОРИЯ И ПРАКТИКА
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Yensen tengsizligi va uning tadbiqlari.

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1.Kirish.

Ma'lumki tengsizliklar matematikada katta o'rin egallaydi. Biz ushbu maqolada tengsizliklarda katta o'rin tutgan Yensen tengsizligi haqida yozdik.Shuningdek Yensen tengsizligidan foydalanib bir nechta mashhur tengsizliklarni isbotlash va bir nechta misollarga tadbiqini kiritdik. Bu maqola asosan matematika bilan shug'llanuvchi va xalqaro olimpiadalarga tayyorlanuvchi o'quvchilar uchun mo'ljallangan. Biz daslab Yensen tengsizligiga kirishdan oldin biz uchun muhim bo'lgan funkiyaning qavariqligi haqida biroz ma'lumotlarni ko'rib chiqamiz.

2.Funksiyaning qavariqligi va botiqligi.

Faraz qilaylik $f(x)$ funksiya (a,b) intervalda berilgan bo'lib, $x_1, x_2 \in (a,b)$ uchun $x_1 < x_2$ bo'lsin.

$f(x)$ funskiya grafigining $(x_1, f(x_1)), (x_2, f(x_2))$ nuqtalaridan o'tuvchi to'g'ri chiziqni $y = l(x)$ quyidagicha

$$l(x) = \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$$

bo'ladi.

1-ta'rif. Agar har qanday oraliq $(x_1, x_2) \subset (a,b)$ intervalda joylashgan $\forall x \in (x_1, x_2)$ uchun

$$f(x) \le l(x)$$

bo'lsa, $f(x)$ funksiya (a,b) intervalda quyidan qavariq funksiya deyiladi. Agar $f(x) < l(x)$ tengsizlik bajarilsa $f(x)$ funksiya (a,b) intervalda quyidan qat'iy qavariq funksiya deyiladi.

2-ta'rif. Agar har qanday oraliq $(x_1, x_2) \subset (a,b)$ intervalda joylashgan $\forall (x) \in (x_1, x_2)$ uchun

$$f(x) \geq l(x)$$

bo'lsa, $f(x)$ funksiya (a,b) intervalda yuqoridan qavariq funksiya deyiladi. Agar $f(x) > l(x)$ tengsizlik bajarilsa $f(x)$ funksiya (a,b) intervalda yuqoridan qat'iy qavariqfunksiya deyiladi.

Funksiyaning quyidan va yuqoridan qavaraqligini quyidagicha ham ta'riflash hammumkin.

Aytaylik, $\alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_1 + \alpha_2 = 1$ bo'lib, $\forall x_1, x_2 \in (a,b)$ bo'lsin.

3-ta'rif. Agar

$$f(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

bo'lsa, $f(x)$ funksiya (a, b) intervalda quyidan qavariq funksiya deyiladi. Agar ishora qa'tiy kichik bo'lsa, $f(x)$ funksiya (a, b) intervalda quyidan qat'iy qavariq funksiya deyiladi.

4-ta'rif. Agar

$$f(\alpha_1 x_1 + \alpha_2 x_2) \geq \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

bo'lsa, $f(x)$ funksiya (a, b) intervalda yuqorida qavariq funksiya deyiladi. Agar ishora qat'iy katta bo'lsa, $f(x)$ funksiya (a, b) intervalda yuqoridan qat'iy qavariq funksiya deyiladi.

Endi bir nechta teoremani isbotsiz keltirib o'tamiz.

1-teorema. Faraz qilaylik, $f(x)$ funksiya (a, b) intervalda berilgan bo'lib, unda $f'(x)$ hosilaga ega bo'lsin. $f(x)$ funksiyaning (a,b) intervalda quyidan qavariq bo'lishi uchun $f'(x)$ ning (a,b) intervalda o'suvchi bo'lishi zarur va yetarli.

2-teorema. Faraz qilaylik, $f(x)$ funksiya (a, b) intervalda berilgan bo'lib, unda $f'(x)$ hosilaga ega bo'lsin. $f(x)$ funksiyaning (a,b) intervalda yuqoridan qavariq bo'lishi uchun $f'(x)$ ning (a,b) intervalda kamayuvchi bo'lishi zarur va yetarli.

Aytaylik, $f(x)$ funksiya $(a,\ b)$ intervalda berilgan bo'lib, u shu intevalda $f''(x)$ hosilaga ega bo'lsin. Bundan tashqari, (a, b) intervalning har qanay $(\alpha,\ \beta)$ $\big((\alpha,\ \beta)\subset(a,\ b)\big)$ qismida $f''(x)$ aynan nolga teng bo'lmasin.

3-teorema. $f(x)$ funksiyaning $(a,\ b)$ intervalda quyidan qavariq bo'lishi uchun $(a,\ b)$ intevalda $f''(x)\geq 0$ bo'lishi zarur va yetarli.

4-teorema. $f(x)$ funksiyaning $(a,\ b)$ intervalda yuqoridan qavariq bo'lishi uchun $(a,\ b)$ intevalda $f''(x)\leq 0$ bo'lishi zarur va yetarli.

Biz Yensen tengsizligini ko'rganimizda asosan 3 va 4-teoremalardan foydalanamiz.

3.Yensen tengsizligi.

Yensen tengsizligi. $f:\ (a,\ b)\to\mathbb{R}$ funsiya $(a,\ b)$ intervalda quyidan qavariq bo'lsin. U holda $n\in\mathbb{N}$ barcha $x_1,\ x_2,\ ...,\ x_n\in(a,\ b)$ lar va $\lambda_1+\lambda_2+...+\lambda_n=1$ shartni qanoatlantiruvchi ixtiyoriy $\lambda_1,\lambda_2,...,\lambda_n\in(0,1)$ sonlar uchun

$$f(\lambda_1x_1+\lambda_2x_2+...+\lambda_nx_n)\leq\lambda_1f(x_1)+\lambda_2f(x_2)+...\lambda_nf(x_n)$$

tengsizlik o'rinli.

Isbot. Biz bu tengsizlikni isbotlashda induksiya metodidan foydalanamiz.

$n=1$ da $\lambda_1=1$ bo'ladi va $f(x_1)=f(x_1)$ ekanligida biz $f(\lambda_1x_1)=\lambda_1f(x_1)$ tenglikka ega bo'lamiz. Shuning uchun tengsizlik o'rinli.

Agar $n=2$ bo'lsa tengsizlik qavariq funksiyaning ta'rifiga to'g'ri keladi.

Faraz qilaylik $n=k$ da barcha $x_1,x_2,...,x_k\in(a,b)$ lar va $\lambda_1+\lambda_2+...+\lambda_k=1$ shartni qanoatlantiruvchi ixtiyoriy $\lambda_1,\lambda_2,...,\lambda_k\in(0,1)$ sonlar uchun

$$f(\lambda_1x_1+\lambda_2x_2+...+\lambda_kx_k)\leq\lambda_1f(x_1)+\lambda_2f(x_2)+...\lambda_kf(x_k)$$

tengsizlik o'rinli bo'lsin.

$n=k+1$ da barcha $x_1,x_2,...,x_{k+1}\in(a,b)$ lar va $\lambda_1+\lambda_2+...+\lambda_{k+1}=1$ shartni qanoatlantiruvchi ixtiyoriy $\lambda_1,\lambda_2,...,\lambda_{k+1}\in(0,1)$ sonlar bo'lsin. U holda

$$\lambda_1x_1+\lambda_2x_2+...+\lambda_{k+1}x_{k+1}=\left(\lambda_1x_1+\lambda_2x_2+...+\lambda_kx_k\right)+\lambda_{k+1}x_{k+1}=$$

$$=\left(1-\lambda_{k+1}\right)\left(\frac{\lambda_1}{1-\lambda_{k+1}}x_1+\frac{\lambda_2}{1-\lambda_{k+1}}x_2+...+\frac{\lambda_k}{1-\lambda_{k+1}}x_k\right)+\lambda_{k+1}x_{k+1} \qquad (1)$$

Agar

$$y_{k+1}=\frac{\lambda_1}{1-\lambda_{k+1}}x_1+\frac{\lambda_2}{1-\lambda_{k+1}}x_2+...+\frac{\lambda_k}{1-\lambda_{k+1}}x_k$$

deb belgilasak.U holda $x_1,x_2,...,x_k\in(a,b)$ ligidan

$$y_{k+1}=\frac{\lambda_1}{1-\lambda_{k+1}}x_1+\frac{\lambda_2}{1-\lambda_{k+1}}x_2+...+\frac{\lambda_k}{1-\lambda_{k+1}}x_k$$

$$>\frac{\lambda_1}{1-\lambda_{k+1}}a+\frac{\lambda_2}{1-\lambda_{k+1}}a+...+\frac{\lambda_k}{1-\lambda_{k+1}}a=\frac{\lambda_1+\lambda_2+...+\lambda_k}{1-\lambda_{k+1}}a=a$$

Shunga o'xshash $y_{k+1}<b$ ekanligi isbotlanadi. Demak $y_{k+1}\in(a,b)$ bo'ladi. Quyidan qavariq funksiya ta'rifi va (1) dan foydalansak, biz quyidagiga ega bo'lamiz.

$$f\left(\lambda_1x_1+\lambda_2x_2+...+\lambda_{k+1}x_{k+1}\right)=f((1-\lambda_{k+1})y_{k+1}+\lambda_{k+1}x_{k+1})$$

$$\le(1-\lambda_{k+1})f\left(y_{k+1}\right)+\lambda_{k+1}f\left(x_{k+1}\right) \qquad (2)$$

bizga ma'lumki.

$$\frac{\lambda_1}{1-\lambda_{k+1}}+\frac{\lambda_2}{1-\lambda_{k+1}}+...+\frac{\lambda_k}{1-\lambda_{k+1}}=1$$

tenglik o'rinli. U holda

$$f(y_{k+1}) = f(\frac{\lambda_1}{1-\lambda_{k+1}}x_1 + \frac{\lambda_2}{1-\lambda_{k+1}}x_2 + ... + \frac{\lambda_k}{1-\lambda_{k+1}}x_k)$$

$$\leq \frac{\lambda_1}{1-\lambda_{k+1}}f(x_1) + \frac{\lambda_2}{1-\lambda_{k+1}}f(x_2) + ... + \frac{\lambda_k}{1-\lambda_{k+1}}f(x_k) \qquad (3)$$

va ninoyat, (2) va (3) dan biz isbotlayotgan

$$f\left(\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_{k+1}x_{k+1}\right) \leq \lambda_1 f\left(x_1\right) + \lambda_2 f\left(x_2\right) + ...\lambda_{k+1} f\left(x_{k+1}\right)$$

tengsizlik kelib chiqadi. Demak matematik induksiya metodidan foydalanib Yensen tengsizligi isbotlandi.

Eslatma. Agar f quyidan qa'tiy qavariq funksiya bo'lsa, Yensen tengsizligining tenglik holati faqat $x_1 = x_2 = ... = x_n$ bo'lganda bajariladi.

Agar f funksiya yuqoridan qavariq bo'lsa, Yensen tengsizligida faqat tengsizlik ishorasi o'zgaradi. Ya'ni

$$f(\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n) \geq \lambda_1 f\left(x_1\right) + \lambda_2 f\left(x_2\right) + ... + \lambda_n f\left(x_n\right)$$

Yensen tengsizligini yana quyidagich ham yozish mumkin.

Agar $f: I \to \mathbb{R}$ funsiya I da quyidan qavariq, $x_1, x_2, ..., x_n \in I$ va $m_1 + m_2 + ... + m_n > 0$ shartni qanoatlantiruvchi $m_1, m_2, ..., m_n \geq 0$ haqiqiy sonlar uchun

$$f\left(\frac{m_1 x_1 + m_2 x_2 + ... + m_n x_n}{m_1 + m_2 + ... + m_n}\right) \leq \frac{m_1 f\left(x_1\right) + m_2 f\left(x_2\right) + ... + m_n f\left(x_n\right)}{m_1 + m_2 + ... + m_n}$$

tengsizlik o'rinli.

4. Yensen tengsizligidan foydalanib mashhur tengsizliklarni isbotlash.

AM-GM tengsizligi. Agar $x_1, x_2, ..., x_n$ musbat haqiqy sonlar uchun

$$\sqrt[n]{x_1 x_2 ... x_n} \leq \frac{x_1 + x_2 + ... + x_n}{n}$$

Isbot. Agar $f(x) = -\ln x$ funsiya $(0,\infty)$ intervalda qaraylik. Bu funsiyaningx 2-tartibli hosilasi $f''(x) = \frac{1}{x^2} > 0$ bo'ladi, demak $f(x)$ quyidan qa'tiy qavariq funksiya $(0,\infty)$ intervalda.

$\lambda_1 = \lambda_2 = \ldots = \lambda_n = \frac{1}{n}$ va $x_i \in (0,\infty), i = 1,2,\ldots,n$ ucun Yensen tengsizligini qo'llaymiz.

$$-\ln\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right) \le -\left(\frac{\ln(x_1) + \ln(x_2) + \ldots + \ln(x_n)}{n}\right)$$

$$\Leftrightarrow \frac{\ln(x_1) + \ln(x_2) + \ldots + \ln(x_n)}{n} \le \ln\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)$$

$$\Leftrightarrow \ln(x_1 x_2 \ldots x_n)^{1/n} \le \ln\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)$$

bundan

$$\sqrt[n]{x_1 x_2 \ldots x_n} \le \frac{x_1 + x_2 + \ldots + x_n}{n}$$

$AM - GM$ tengsizligi kelib chiqadi.

Teorema.(Koshi-Bunyokovskiy tengsizligi) Agar $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n$ haqiqiy sonlar uchun

$$(a_1 b_1 + a_2 b_2 + \ldots a_n b_n)^2 \le (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2)$$

tengsizlik o'rinli va tenglik holati faqat va faqat $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n}$ bo'lganda bo'ladi.

Isbot. Agar $f(x) = x^2$ funksiyani $\mathbb{R}$ sohada qarasak. $f''(x) = 2 > 0$ bo'ladi. Ya'ni $f(x)$ funksiyau quyidan qavariq bo'ladi. U holda Yensen tengsizligidan foydalansak

$$f\left(\frac{m_1x_1+m_2x_2+...+m_nx_n}{m_1+m_2+...+m_n}\right)\leq\frac{m_1f(x_1)+m_2f(x_2)+...+m_nf(x_n)}{m_1+m_2+...+m_n}.$$

Biz quyidagini olamiz

$$\left(\frac{m_1x_1+m_2x_2+...+m_nx_n}{m_1+m_2+...+m_n}\right)^2\leq\frac{m_1x_1^2+m_2x_2^2+...+m_nx_n^2}{m_1+m_2+...+m_n}$$

yoki

$$(m_1x_1+m_2x_2+...+m_nx_n)^2\leq(m_1x_1^2+m_2x_2^2+...+m_nx_n^2)(m_1+m_2+...+m_n)$$

Agar $i=1,2,...n$ uchun $x_i=\frac{a_i}{b_i}, m_i=b_i^2$ deb belgilash kiritsak.

U holda biz quyidagi

$$(a_1b_1+a_2b_2+...a_nb_n)^2\leq(a_1^2+a_2^2+...+a_n^2)(b_1^2+b_2^2+...+b_n^2)$$

Koshi-Bunyakovskiy tengsizligiga ega bo'lamiz.

Teorema.(Gyuygens tengsizligi) Agar $a_1,a_2,...,a_n$ musbat haqiqiy sonlar uchun

$$(1+a_1)(1+a_2)...(1+a_n)\geq(1+\sqrt[n]{a_1a_2...a_n})^n$$

tengsizlik o'rinli.

Isbot. Agar $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\ln(1+e^x)$ funksiyani qarasak. Barcha $x\in\mathbb{R}$ uchun $f''(x)=\frac{e^x}{(1+e^x)^2}>0$ tenglik o'rinli bo'ladi. Demak $f(x)$ funksiya quyidan qavariq bo'ladi.Yensen tengsizligidan foydalanamiz.

$$f\left(\frac{m_1x_1+m_2x_2+...+m_nx_n}{m_1+m_2+...+m_n}\right)\leq\frac{m_1f(x_1)+m_2f(x_2)+...+m_nf(x_n)}{m_1+m_2+...+m_n}.$$

Agar $i=1,2,...n$ uchun $m_i=\dfrac{1}{n}$ va $x_i=\ln a_i$ deb almashtirish bajarsak,

$$f\left(\frac{\ln a_1+\ln a_2+...+\ln a_n}{n}\right)\leq\frac{f(\ln a_1)+f(\ln a_2)+...+f(\ln a_n)}{n}$$

$$\Leftrightarrow \ln\left(1+\exp\left(\frac{\ln a_1a_2...a_n}{n}\right)\right)\leq\frac{\ln(1+a_1)+\ln(1+a_2)+...+\ln(1+a_n)}{n}$$

$$\Leftrightarrow n\ln(1+\sqrt[n]{a_1a_2...a_n})\leq\ln(1+a_1)(1+a_2...(1+a_n))$$

$$\Leftrightarrow (1+a_1)(1+a_2)...(1+a_n)\geq(1+\sqrt[n]{a_1a_2...a_n})^n$$

bu esa Gyuygens tengsizligi.

Teorema.(Yung tengsizligi) Agar $p,q>1$ haqiqiy sonlar $\dfrac{1}{p}+\dfrac{1}{q}=1$ tenglikni qanoatlantirsa. u holda ixtiyoriy $a,b>0$ uchun $ab\leq\dfrac{a^p}{p}+\dfrac{b^q}{q}$ tengsizli o'rinli va tenglik holati faqat va faqat $a^p=b^q$ bo'ladi.

Isbot. Agar $f(x)=e^x$ funksiyani $(0,\infty)$ sohada qarasak $f''(x)=e^x>0$ bo'ladi ixtiyoriy $x\in\mathbb{R}$. Demak $f(x)$ funksiya $(0,\infty)$ sohada quyidan qavariq. Yensen tengsizligidan $n=2$ bo'lgan holidan foydalanamiz.

$$f\left(\frac{m_1x_1+m_2x_2}{m_1+m_2}\right)\leq\frac{m_1f(x_1)+m_2f(x_2)}{m_1+m_2}$$

tengsizlikda $m_1=\dfrac{1}{p},m_2=\dfrac{1}{q},x_1=p\ln a$ va $x_2=q\ln b$ belgilask kiritsak.

$$f\left(\frac{x_1}{p}+\frac{x_2}{q}\right)\leq\frac{1}{p}f(x_1)+\frac{1}{q}f(x_2)\Leftrightarrow e^{\frac{x_1}{p}+\frac{x_2}{q}}\leq\frac{e^x}{p}+\frac{e^y}{q}$$

$$\Leftrightarrow e^{\ln a+\ln b}\le \frac{e^{p\ln a}}{p}+\frac{e^{q\ln b}}{q}\Leftrightarrow e^{\ln ab}\le \frac{e^{\ln a^p}}{p}+\frac{e^{\ln b^q}}{q}$$

$$\Leftrightarrow ab\le \frac{a^p}{p}+\frac{b^q}{q}$$

tensizlikga ega bo'lamiz. Tenglik holati $x_1=x_2$ bo'lganda,ya'ni, $a^p=b^q$ holda bo'ladi. Yung tengsizligi isbotlandi.

Teorema.(Gyolder tengsizligi) Agar $a_1,a_2,...,a_n;b_1,b_2,...,b_n$ musbat haqiqiy sonlar va $\frac{1}{p}+\frac{1}{q}=1$ sharni qanoatlantiruvchi $p,q>1$ sonlar uchun

$$\sum_{i=1}^{n}a_ib_i\le\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}b_i^q\right)^{\frac{1}{q}}$$

tengsizlik o'rinli va tenglik holati faqat va faqat $\frac{a_1^p}{b_1^q}=\frac{a_2^p}{b_2^q}=...=\frac{a_n^p}{b_n^q}$ da o'rinli.

Isbot. $f:\mathbb{R}^+\to\mathbb{R}, f(x)=x^p$ funsiya $p>1$ uchun qaraymiz. $f(x)$ funksiya quyidan qavariq chunki $f''(x)=p(p-1)x^{p-2}>0$ bo'ladi. U xolda Yensen tengsizligigni qo'llasak.

$$f\left(\frac{m_1x_1+m_2x_2+...+m_nx_n}{m_1+m_2+...+m_n}\right)\le\frac{m_1f(x_1)+m_2f(x_2)+...+m_nf(x_n)}{m_1+m_2+...+m_n}$$

$$\left(\frac{m_1x_1+m_2x_2+...+m_nx_n}{m_1+m_2+...+m_n}\right)^p\le\frac{m_1x_1^p+m_2x_2^p+...+m_nx_n^p}{m_1+m_2+...+m_n}.$$

yoki

$$(m_1x_1+m_2x_2+...+m_nx_n)^p\le(m_1+m_2+...+m_n)^{p-1}(m_1x_1^p+m_2x_2^p+...+m_nx_n^p)$$

$$\Leftrightarrow m_1x_1+m_2x_2+...+m_nx_n\le(m_1+m_2+...+m_n)^{\frac{p-1}{p}}(m_1x_1^p+m_2x_2^p+...+m_nx_n^p)^{\frac{1}{q}}$$

$\frac{1}{p}+\frac{1}{q}=1$ tenglikdan $\frac{p-1}{p}=\frac{1}{q}$ tenglik kelib chiqadi. Bu tenglikni

yuqoridagi tengsizlikga qo'ysak.

$$\sum_{i=1}^{n} m_i x_i \le \left(\sum_{i=1}^{n} m_i\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} m_i x_i^p\right)^{\frac{1}{p}}$$

$i=1,2,...,n$ uchun $m_i = b_i^q$ va $x_i = a_i b_i^{1-q}$ deb belgilash kiritsak. Biz

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}.$$

Gyolder tengsizligiga ega bo'lamiz.

Eslatma. Agar $p=q=2$ bo'lsa Gyolder tengsizligidan Koshi-Bunyokovskiy tengsizligi kelib chiqadi.

Teorema.(Minkovskiyning ikkinchi tengsizligi) Agar $a_1, a_2, ..., a_n; b_1, b_2, ..., b_n$ musbat haqiqiy sonlar va $p>1$ sonlar uchun

$$\left(\left(\sum_{i=1}^{n} a_i\right)^p + \left(\sum_{i=1}^{n} b_i\right)^p\right)^{\frac{1}{q}} \le \sum_{i=1}^{n} (a_i^p + b_i^q)^{\frac{1}{q}}$$

tensizli o'rinli va tenglik holadi faqat va faqat $\frac{a_1}{b_1}=\frac{a_2}{b_2}=...=\frac{a_n}{b_n}$ bo'lganda bo'ladi.

Isbot. Agar $f:\mathbb{R}^+ \to \mathbb{R}^+, f(x)=(1+x^p)^{\frac{1}{p}}$ funksiya $p>1$ da quyidan qa'tiy qavariq chunki $f''(x)=(p-1)(x^{-p}+1)^{\frac{1-2p}{p}} \cdot x^{-p-1} > 0$ ixtiyoriy $x \in \mathbb{R}^+$ va $p>1$ lar uchun.Yensen tengsizligidan foydalansak.

$$\left(1+\left(\frac{m_1x_1+m_2x_2+...+m_nx_n}{m_1+m_2+...+m_n}\right)^p\right)^{\frac{1}{p}}$$

$$\leq\frac{(m_1(1+x_1^p)^{\frac{1}{p}}+m_2(1+x_2^p)^{\frac{1}{p}}+...+m_n(1+x_n^p)^{\frac{1}{p}}}{m_1+m_2+...+m_n}$$

$$\Leftrightarrow\left(\left(\sum_{i=1}^n m_i\right)^p+\left(\sum_{i=1}^n m_ix_i\right)^p\right)^{\frac{1}{p}}\leq\sum_{i=1}^n((m_i)^p+(m_ix_i)^p)^{\frac{1}{p}}$$

Agar biz $i=1,2,...,n$ uchun $m_i=a_i$ va $x_i=\frac{b_i}{a_i}$ almashtirish bajarsak

$$\left(\left(\sum_{i=1}^n a_i\right)^p+\left(\sum_{i=1}^n b_i\right)^p\right)^{\frac{1}{q}}\leq\sum_{i=1}^n(a_i^p+b_i^q)^{\frac{1}{q}}$$

tegsizliga ega bo'lamiz. Tengsizlik isbotlandi.

Teorema.(Minkoviskiyning uchinchi tengsizligi) Agar $a_1,a_2,...,a_n;b_1,b_2,...,b_n$ musbat haqiqiy sonlar uchun

$$\sqrt[n]{a_1a_2...a_n}+\sqrt[n]{b_1b_2...b_n}\leq\sqrt[n]{(a_1+b_1)(a_2+b_2)...(a_n+b_n)}$$

tengsizlik o'rinli va tenglik holati faqat va faqat $\frac{a_1}{b_1}=\frac{a_2}{b_2}=...=\frac{a_n}{b_n}$ bo'lganda barajiriladi.

Isbot. Agar $f(x)=\ln(1+e^x)$ funksiyani $\mathbb{R}$ da quyidan qavariq chunki $f''(x)=\frac{e^x}{(1+e^x)^2}>0$ ixtiyoriy $x\in\mathbb{R}$ uchun. Yensin tensizligidan foydalansak

$$f\left(\frac{m_1x_1+m_2x_2+...+m_nx_n}{m_1+m_2+...+m_n}\right)\leq\frac{m_1f(x_1)+m_2f(x_2)+...+m_nf(x_n)}{m_1+m_2+...+m_n}$$

tengsizlikda $i=1,2,...,n$ uchun $m_i=\frac{1}{n}$ va $x_i=\ln\frac{b_i}{a_i}$ almashtirish bajaramiz.

$$\ln\left(1+\exp(\frac{1}{n}\sum_{i=1}^{n}\ln\frac{b_i}{a_i})\right)\le\frac{1}{n}\sum_{i=1}^{n}\ln\left(1+\exp(\ln\frac{b_i}{a_i})\right)$$

$$\Leftrightarrow\ln\left(1+\prod_{i=1}^{n}\sqrt[n]{\frac{b_i}{a_i}}\right)\le\frac{1}{n}\ln\prod_{i=1}^{n}(1+\frac{b_i}{a_i})$$

$$\Leftrightarrow 1+\sqrt[n]{\frac{b_1b_2...b_n}{a_1a_2...a_n}}\le\sqrt[n]{(1+\frac{b_1}{a_1})(1+\frac{b_2}{a_2})...(1+\frac{b_n}{a_n})}$$

tengsizlikning ikkala tomonini $\sqrt[n]{a_1a_2...a_n}$ ga ko'paytirsak

$$\sqrt[n]{a_1a_2...a_n}+\sqrt[n]{b_1b_2...b_n}\le\sqrt[n]{(a_1+b_1)(a_2+b_2)...(a_n+b_n)}$$

tengsizlik kelib chiqadi. Minkovskiy tengsizligi isbotlandi.

5.Yensen tengsizligining tadbiqlari.

1-misol. ABC uchburchakning α,β va γ burchaklari uchun quyidagi tengsizlikni isbotlang.

$$\sin\alpha\sin\beta\sin\gamma\le\frac{3\sqrt{3}}{8}$$

Isbot. Bizga malumki $\alpha,\beta,\gamma\in(0,\pi)$ va $\sin\alpha,\sin\beta,\sin\gamma>0$ bo'ladi. U holda $AM-GM$ tengsizligidan foydalanib

$$\sqrt[3]{\sin\alpha\sin\beta\sin\gamma}\le\frac{\sin\alpha+\sin\beta+\sin\gamma}{3}$$

ga ega bo'lamiz.

Endi $f(x)=\sin x$ funksiya $(0,\pi)$ da yuqoridan qavariq chunki $f''(x)=-\sin x<0$ bo'ladi barcha $x\in(0,\pi)$ uchun. Yensin tengsizligidan foydalanamiz.

$$\frac{\sin\alpha+\sin\beta+\sin\gamma}{3}\le\sin\left(\frac{\alpha+\beta+\gamma}{3}\right)=\frac{\sqrt{3}}{2}$$

U holda

$$\sqrt[3]{\sin\alpha\sin\beta\sin\gamma}\le\frac{\sin\alpha+\sin\beta+\sin\gamma}{3}=\frac{\sqrt{3}}{2}$$

$$\Leftrightarrow\sin\alpha\sin\beta\sin\gamma\le\frac{3\sqrt{3}}{8}$$

tengsizlik hosil bo'ladi.Tengsizlik isbotlandi.

2-misol. Agar $x>0,y>0,z>0$ va $x+y+z=1$ bo'lsa quyidagi tengsizlikni isbotlang.

$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right)\ge 64$$

Isbot. Agar $f(t)=\ln(1+\frac{1}{t})$ funksiyani $(0,\infty)$ sohada qarasak. f funksiya quyidan qavariq chunki $f''(t)=\frac{1}{t^2}-\frac{1}{(1+t)^2}>0$ bo'ladi ixtiyoriy $x\in(0,\infty)$ uchun. $n=3$ holida Yensen tengsizlikini qo'llasak.

$$f(\lambda_1x_1+\lambda_2x_2+\lambda_3x\ )\le\lambda_1f(x_1)+\lambda_2f(x_2)+\lambda_3f(x_3).$$

Agar $\lambda_1=\lambda_2=\lambda_3=\frac{1}{3},x_1=x,x_2=y,x_3=z$ deb belgilash kiritamiz

$$f\left(\frac{x}{3}+\frac{y}{3}+\frac{z}{3}\right)\le\frac{1}{3}(f(x)+f(y)+f(z))$$

$$\Leftrightarrow \ln(1+\frac{1}{x})+\ln(1+\frac{1}{y})+\ln(1+\frac{1}{z})\geq 3\ln(1+\frac{3}{x+y+z})$$

$$\Leftrightarrow \ln(1+\frac{1}{x})(1+\frac{1}{y})(1+\frac{1}{z})\geq 3\ln 4$$

$$\Leftrightarrow (1+\frac{1}{x})(1+\frac{1}{y})(1+\frac{1}{z})\geq 64$$

kelib chiqadi.Tengsizlik isbotlandi.

3-misol. $a_1,a_2,...,a_n>1$ haqiqiy sonlar uchun

$$(a_1a_2...a_n)^{\sqrt[n]{a_1a_2...a_n}}\leq a_1^{a_1}a_2^{a_2}...a_n^{a_n}$$

tengsizlikni isbotlang.

Isbot. Agar $f(x)=xe^x$ funksiya $(0,\infty)$ sohada qaraymiz. f funksiya quyidan qavariq chunki $f''(x)=2e^x+xe^x>0$ bo'ladi ixtiyoriy $x\in(0,\infty)$ uchun. Yensen tengsizligini qo'llaymiz.

$$f\left(\frac{m_1x_1+m_2x_2+...+m_nx_n}{m_1+m_2+...+m_n}\right)\leq\frac{m_1f(x_1)+m_2f(x_2)+...+m_nf(x_n)}{m_1+m_2+...+m_n}$$

Agar $m_1=m_2=...=m_n=1$ deb belgilasak

$$f\left(\frac{x_1+x_2+...+x_n}{n}\right)\leq\frac{f(x_1)+f(x_2)+...+f(x_n)}{n}.$$

$$\Leftrightarrow (x_1+x_2+...+x_n)e^{\frac{x_1+x_2+...+x_n}{n}}\leq x_1e^{x_1}+x_2e^{x_2}+...+x_ne^{x_n}$$

Agar $i=1,2,...,n$ uchun $x_i=lna_i$ deb belgilsh kiritsak va soddalashtirsak.

$$\ln(a_1a_2...a_n)e^{\frac{\ln(a_1a_2...a_n)}{n}}\leq a_1\ln a_1+a_2\ln a_2...+a_n\ln a_n$$

$$\Leftrightarrow \ln(a_1a_2...a_n)(a_1a_2...a_n)^{\frac{1}{n}}\leq\ln(a_1^{a_1}a_2^{a_2...a_n^{a_n}})$$

$$\Leftrightarrow (a_1 a_2 \ldots a_n)^{\sqrt[n]{a_1 a_2 \ldots a_n}} \le a_1^{a_1} a_2^{a_2} \ldots a_n^{a_n}$$

tengsizlik kelib chiqadi. Tengsizlik isbotlandi.

4-misol. Agar $a_1, a_2, \ldots, a_n$ musbat haqiqiy sonlar bo'lsin u holda k ning qanday qiymatida quyidagi tengsizlik o'rinli bo'ladi.(bu yerda k soni n ga bog'liq)

$$\sqrt[n]{a_1} + \sqrt[n]{a_2} \ldots + \sqrt[n]{a_n} \le k \sqrt[n]{a_1 + a_2 + \ldots + a_n}$$

Yechish. Agar $f : \mathbb{R}^+ \to \mathbb{R}, f(x) = x^p$ funsiya $0 < p < 1$ uchun qaraymiz. $f(x)$ funksiya yuqoridan qavariq chunki $f''(x) = p(p-1)x^{p-2} < 0$ bo'ladi. U xolda Yensen tengsizligigni qo'llasak.

$$\left(\frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{m_1 + m_2 + \ldots + m_n} \right)^p \ge \frac{m_1 x_1^p + m_2 x_2^p + \ldots + m_n x_n^p}{m_1 + m_2 + \ldots + m_n}.$$

yoki biroz soddalashtirsak

$$(m_1 x_1 + m_2 x_2 + \ldots + m_n x_n)^p \ge (m_1 + m_2 + \ldots + m_n)^{p-1} (m_1 x_1^p + m_2 x_2^p + \ldots + m_n x_n^p)$$

ko'rinishga keladi. Agar $i = 1, 2, \ldots, n$ uchun $x_i = a_i$, $m_1 = m_2 = \ldots = m_n = \frac{1}{n}$, va $p = \frac{1}{n}$

deb belgilash kiritsak

$$\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^{\frac{1}{n}} \ge \frac{a_1^{\frac{1}{n}} + a_2^{\frac{1}{n}} + \ldots + a_n^{\frac{1}{n}}}{n}$$

$$\Leftrightarrow \sqrt[n]{a_1} + \sqrt[n]{a_2} \ldots + \sqrt[n]{a_n} \le n^{\frac{n-1}{n}} \sqrt[n]{a_1 + a_2 + \ldots + a_n}$$

bo'ladi. Demak $k = n^{\frac{n-1}{n}}$ ekan.

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