

BETA VA GAMMA FUNKSIYA VA UNING TADBIQLARI
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Fizika-Matematika fakulteti

Matematika ta'lim yo'nalishi 3-kurs

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Fizika-matematika fanlari bo'yicha falsafa doktori.

Annotatsiya: Ushbu maqolada Beta hamda Gamma funksiya hamda ularning tadbiqlarini o'rganib chiqamiz.

Kalit so'zlar: Beta funksiya, Gamma funksiya, integral, Eyler integrali, I tur Eyler integrali, II tur Eyler integrali.

Beta funksiya [I tur Eyler integrali].

1.1-Tarif.

$$\int_0^1 x^{a-1}(1-x)^{b-1} dx$$

integral *beta funksiy* yoki *I tur Eyler integrali* deb ataladi va $B(a, b)$ kabi belgilanadi. De-mak

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx \quad (a > 0, b > 0)$$

bo'ladi.

Gamma funksiya [II tur Eyler integrali].

1.2-Tarif.

$$\int_0^{+\infty} x^{a-1} e^{-x} dx$$

integral *gamma funksiya* yoki *II tur Eyler integrali* deb ataladi va $\Gamma(a)$ kabi belgilanadi. Demak

$$\Gamma(a) = \int_0^{+\infty} x^{a-1} e^{-x} dx \quad (0, +\infty)$$

bo'ladi.



Biz quyida $B(a, b)$ va $\Gamma(a)$ funksiyalar orasidagi bog'lanishni ifodalaydigan formulani keltiramiz.

Ma'lumki, $\Gamma(a)$ funksiya $(0, +\infty)$ da, $B(a, b)$ funksiya esa R^2 fazodagi $M = \{(a, b) \in R^2 : a \in (0, +\infty), b \in (0, +\infty)\}$ to'plam berilgan.

1.1– Teorema. $\forall (a, b) \in M$ uchun

$$B(a, b) = \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a + b)}$$

fo'rmula o'rinlidir.

Isbot.

Ushbu

$$\Gamma(a + b) = \int_0^{+\infty} x^{a+b-1} \cdot e^{-x} dx$$

$(a > 0, b > 0)$ gamma funksiyada o'zgaruvchini quyidagicha almashtiramiz:

$$x = (1 + t)y \quad (t > 0).$$

Natijada quyidagiga ega bo'lamiz:

$$\Gamma(a + b) = \int_0^{+\infty} (1 + t)^{a+b-1} \cdot y^{a+b-1} e^{-(1+t)y} \cdot (1 + t) dy =$$

$$= (1 + t)^{a+b} \int_0^{+\infty} y^{a+b-1} e^{-(1+t)y} dy.$$

Keyingi

tenglikdan

quyidagini

topamiz:

$$\frac{\Gamma(a + b)}{(1 + t)^{a+b}} = \int_0^{+\infty} y^{a+b-1} e^{-(1+t)y} dy$$

Bu tenglikning har ikki tomonini $t^{\alpha-1}$ ga ko'paytirib, natijani $(0, +\infty)$ oraliq bo'yicha integrallaymiz:

$$\Gamma(a + b) \int_0^{+\infty} \frac{t^{a-1}}{(1 + t)^{a+b}} dt = \int_0^{+\infty} \left[\int_0^{+\infty} y^{a+b-1} e^{-(1+t)y} dy \right] t^{a-1} dt.$$





Agar

$$\int_0^{+\infty} \frac{t^{a-1}}{(1+t)^{a+b}} dt = B(a, b) \quad (1)$$

(1) formulaga ko'ra

$$\Gamma(a + b) \cdot B(a, b) = \int_0^{+\infty} \left[\int_0^{+\infty} y^{a+b-1} e^{-(1+t)y} dy \right] t^{a-1} dt. \quad (2)$$

bo'ladi. Endi (2) tenglikning o'ng tomonidagi integral $\Gamma(a) \cdot \Gamma(b)$ ga teng bo'lishini isbotlaymiz. Uning uchun, avvalo bu integrallarda integrallash tartibini almashtirish mumkinligini ko'rsatamiz. Buning uchun biz parametirga bog'liq xosmas integrallarni parameter bo'yicha integralash mavzusidagi (17.19 – teoremadan) foydalanamiz.

17.19-Teorema shartlari bajarilishini ko'rsatishimiz kerak.

Dastlab $a > 1, b > 1$ bo'lgan holni qaraylik.

$a > 1, b > 1$ da, yani $\{(a, b) \in R^2 : a \in (1, +\infty), b \in (1, +\infty)\}$ to'plamda integral ostidagi

$$f(t, y) = y^{a+b-1} t^{a-1} e^{-(1+t)y}$$

funksiya $\forall (t, y) \in \{(t, y) \in R^2 : t \in [0, +\infty), y \in [0, +\infty)\}$ da uzluksiz bo'lib, $f(t, y) = y^{a+b-1} t^{a-1} e^{-(1+t)y} \geq 0$ bo'ladi.

Ushbu

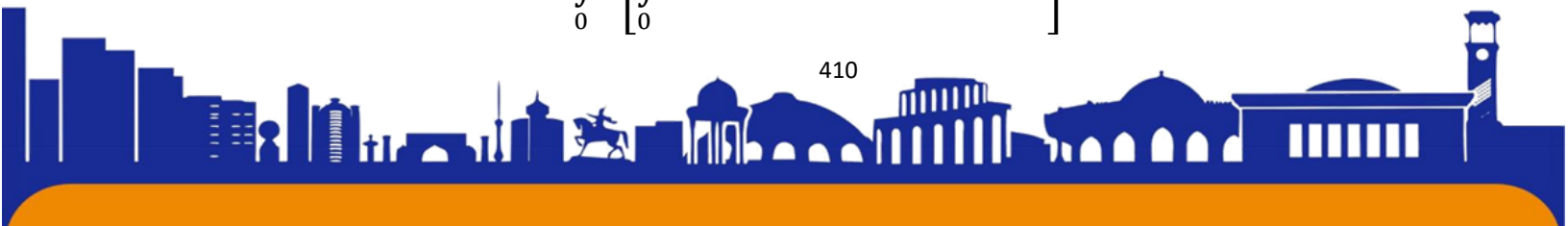
$$\int_0^{+\infty} f(t, y) dt = \int_0^{+\infty} t^{a-1} y^{a+b-1} e^{-(1+t)y} dt$$

integral y o'zgaruvchining $[0, +\infty)$ oraliqdagi uzluksiz funksiyasi bo'ladi, chunki

$$\int_0^{+\infty} t^{a-1} y^{a+b-1} e^{-(1+t)y} dt = \Gamma(a) \cdot y^{b-1} e^{-y}$$

va nihoyat, yuqoridagi (2) munosabatga ko'ra

$$\int_0^{+\infty} \left[\int_0^{+\infty} t^{a-1} y^{a+b-1} e^{-(1+t)y} dy \right] dt$$





integral yaqinlashuvchi kelib chiqadi.

U holda **17.19** Teoreмага asosan

$$\int_0^{+\infty} \left[\int_0^{+\infty} t^{a-1} y^{a+b-1} e^{-(1+t)y} dt \right] dy$$

integral ham yaqinlashuvchi bo'lib,

$$\int_0^{+\infty} \left[\int_0^{+\infty} t^{a-1} y^{a+b-1} e^{-(1+t)y} dy \right] dt = \int_0^{+\infty} \left[\int_0^{+\infty} t^{a-1} y^{a+b-1} e^{-(1+t)y} dt \right] dy$$

bo'ladi. O'ng tomondagi integralni hisoblasak,

$$\begin{aligned} & \int_0^{+\infty} \left[\int_0^{+\infty} t^{a-1} y^{a+b-1} e^{-(1+t)y} dy \right] dt = \\ & = \int_0^{+\infty} y^{b-1} e^{-y} \Gamma(a) dy = \Gamma(a) \cdot \Gamma(b). \quad (3) \end{aligned}$$

tenglik kelib chiqadi.

Natijada (2) va (3) munosabatlardan

$$\Gamma(a + b) \cdot B(a, b) = \Gamma(a) \cdot \Gamma(b)$$

yani

$$B(a, b) = \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a + b)} \quad (4)$$

bo'lishi kelib chiqadi.

2.1 – Natija. $\forall a \in (0, 1)$ uchun

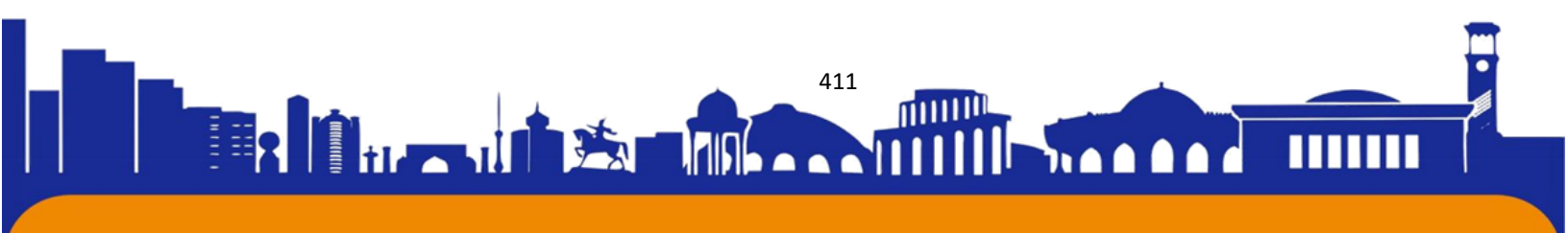
$$\Gamma(a) \cdot \Gamma(1 - a) = \frac{\pi}{\sin a \pi} \quad (6)$$

bo'ladi.

Haqiqattan ham, (4) formula $b = 1 - a$ ($0 < a < 1$) deyilsa, unda

$$B(a, 1 - a) = \frac{\Gamma(a) \cdot \Gamma(1 - a)}{\Gamma(1)}$$

bo'ladi.





Xususan *beta* funksiyada $b = 1 - a$ ($0 < a < 1$) bo'lganda

$$B(a, 1 - a) = \int_0^{+\infty} \frac{t^{a-1}}{1+t} dt = \frac{\pi}{\sin a\pi} \quad (7)$$

(7) munosabat o'rinli bo'ladi. (7) va $\Gamma(1) = 1$ munosabatlarga muvofiq

$$\Gamma(a) \cdot \Gamma(1 - a) = \frac{\pi}{\sin a\pi} \quad (0 < a < 1).$$

(6) formula *keltirish formulasi* deb ataladi.

Misol.

Ushbu

$$\int_0^{\infty} \frac{\sin(x) \ln x}{x} dx$$

integralni hisoblamg.

Yechish.

Integralni

hisoblaymiz

$$\int_0^{\infty} \frac{\sin(x) \ln x}{x} dx \quad [1]$$

[1] integralni

I

bilan

belgilab

olamiz

$$I = \int_0^{\infty} \frac{\sin(x) \ln x}{x} dx \quad [2].$$

Endi

S

parametirga

bog'liq

xosmas

integral

olamiz

$$I(s) = \int_0^{\infty} \frac{\sin(x)}{x^s} dx \quad [3]$$

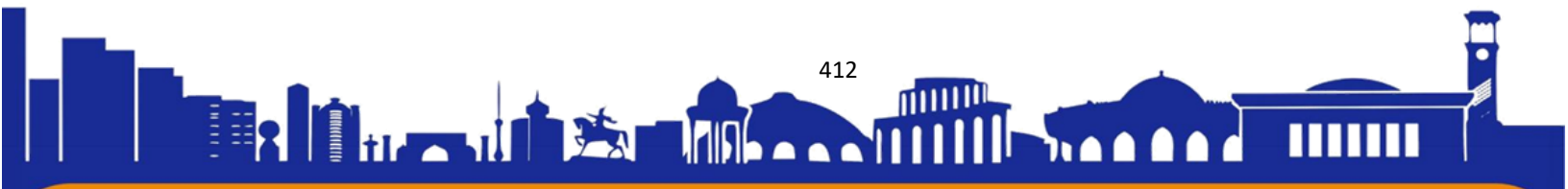
bu integral $S \geq 1$ o'suvchi bo'ladi. [3] integralni S bo'yicha ikkala tomoni ham differen-

sialaymiz

$$\frac{d}{ds} I(s) = \frac{d}{ds} \int_0^{\infty} \frac{\sin(x)}{x^s} dx \quad [4]$$

[4] tenglama hosil bo'ladi. [4] tenglamani yechsak

$$\frac{d}{ds} I(s) = \frac{d}{ds} \int_0^{\infty} \frac{\sin(x)}{x^s} dx = \int_0^{\infty} \frac{2 \sin x}{2s x^s} dx$$



teng bo'ladi. $\frac{2}{2s} x^{-s} = -x^{-s} \ln x$ ga tenglasak

$$I'(s) = - \int_0^{\infty} \frac{\sin x}{x^s} \ln x \, dx \quad [5]$$

[5] tenglamaga kelamiz. Bunda $-I'(s=1) = I$ yani [5] tenglama [2] tenglamaga teng bo'ladi.

Endi [1] integralni yechish uchun [3] integraldan foydalanamiz

$$I(s) = \int_0^{\infty} \frac{\sin(x)}{x^s} \, dx = \int_0^{\infty} \sin x \frac{1}{x^s} \, dx \quad [6]$$

bo'ladi. Gamma funksiyadan foydalanib

$$I_1 = \int_0^{\infty} t^{s-1} e^{-tx} \, dt \quad [7]$$

[7] da $tx = u$ deb t bo'ycha differensialasak va $dt = \frac{1}{x} du$ ni topsak, [7] tenglama

$$I_1 = \int_0^{\infty} \frac{u^{s-1}}{x^{s-1}} \frac{du}{x} = \frac{1}{x^s} \int_0^{\infty} u^{s-1} e^{-u} \, du$$

$$I_1 = \frac{1}{x^s} \int_0^{\infty} u^{s-1} e^{-u} \, du \quad [8]$$

[8] ko'rnishida bo'ladi. Bunda [8] ni chap tomonidagi integral $\Gamma(s)$ teng.

$$\Gamma(s) = \int_0^{\infty} u^{s-1} e^{-u} \, du$$

[8] ni chap tomonidagi integral o'rniga $\Gamma(s)$ ni qo'ysak

$$I_1 = \frac{1}{x^s} \Gamma(s) \quad [9]$$

[9] ga teng bo'ladi. Bunda

$$\frac{1}{x^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-tx} \, dt$$



belgilab, [6] ga quysak

$$I(s) = \int_0^{\infty} \sin x \left\{ \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-tx} dt \right\} dx \quad [10]$$

teng.

$$I(s) = \int_0^{\infty} \sin x \left\{ \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-tx} dt \right\} dx = \frac{1}{\Gamma(s)} \int_0^{\infty} \int_0^{\infty} (\sin x) t^{s-1} e^{-tx} dt dx$$

$$I(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \int_0^{\infty} (\sin x) t^{s-1} e^{-tx} dx dt = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left\{ \int_0^{\infty} (\sin x) e^{-tx} dx \right\} dt$$

bo'laklab integralasak

$$\int_0^{\infty} (\sin x) e^{-tx} dx = \frac{1}{1+t^2}$$

ga teng bo'ladi.

$$I(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{1+t^2} dt$$

bunda $t^2 = y \Rightarrow dt = \frac{1}{2} y^{-1/2} dy$ belgilab integralga quysak

$$I(s) = \frac{1}{2\Gamma(s)} \int_0^{\infty} \frac{y^{\frac{s-1}{2}} y^{-1/2}}{1+y} dy = \frac{1}{2\Gamma(s)} \int_0^{\infty} \frac{y^{\frac{s}{2}-1}}{1+y} dy$$

$\frac{s}{2} = z$ deb

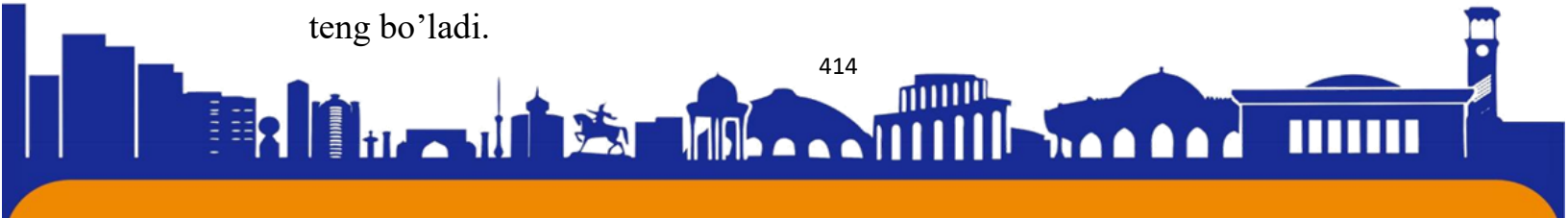
$$\int_0^{\infty} \frac{y^{z-1}}{1+y} dy = \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin z\pi}$$

$$I(s) = \frac{1}{2\Gamma(s)} \frac{\pi}{\sin z\pi} \quad [11]$$

[11] ga teng bo'ladi. Bunda $z = \frac{s}{2}$ [11] ga oborib qo'yamiz

$$I(s) = \frac{1}{2\Gamma(s)} \frac{\pi}{\sin \frac{s\pi}{2}} \quad [12]$$

teng bo'ladi.



Agar $I = -I(s = 1)$ bo'lsa, [12] tenglamadan s bo'yicha hosila olsak

$$I'(s) = \frac{\pi}{2} \left\{ \frac{-\Gamma'(s)}{\Gamma^2(s)} \frac{1}{\sin \frac{s\pi}{2}} - \frac{1}{2\Gamma(s)} \cdot \frac{-\cos \frac{s\pi}{2}}{(\sin \frac{s\pi}{2})^2} \right\}$$

$$I'(1) = \frac{\pi}{2} \left\{ -\Gamma'(1) \frac{1}{\sin \frac{\pi}{2}} - \frac{1}{2} \cdot \frac{-\cos \frac{\pi}{2}}{(\sin \frac{\pi}{2})^2} \right\}$$

$$I'(1) = -\frac{\pi}{2} \Gamma'(1)$$

bunda $-\Gamma'(1) = \gamma$ ga teng

$$I'(1) = -\frac{\pi}{2} \Gamma'(1) = \frac{\pi\gamma}{2}$$

$$\int_0^{\infty} \frac{\sin(x) \ln x}{x} dx = \frac{\pi\gamma}{2}.$$

teng bo'ladi. Misol yechildi.

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