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INVERSE SCATTERING TRANSFORM FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH SELF-CONSISTENT SOURCE

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Abstract. The theory of inverse scattering is studied to solve the initial-value problem for the nonlinear Schrodinger equation with a self-consistent source and the finite density type initial data. In direct scattering problem, we establish the Jost eigenfunctions, scattering data and their analyticity and symmetry. Moreover, the asymptotic behavior of the Jost functions of the Dirac's type operator and scattering data are analyzed needed in inverse problem.

Keywords: nonlinear Schrodinger equation, Zakharov-Shabat system, nonzero boundary condition, inverse scattering theory, self-consistent source.

Introduction.

The theory of nonlinear dynamics has aroused considerable interest and has established a connection with some directions in the field of soliton theory. It is well known that the nonlinear Schrödinger (NLS) equation are some important typical and fully studied nonlinear integrable equation, which can describe a series of nonlinear wave phenomena in dispersive physical structures. One of the examples is the NLS equation

$$iu_t - 2\chi|u|^2 u + u_{xx} = 0,\ \chi = const$$

where the subscripts denote the corresponding partial derivatives, u is the real scalar function with (t, x), and $\chi = \pm 1$ denote the defocusing and focusing NLS equation, respectively.

The NLS equation with various boundary conditions models a wide class of nonlinear phenomena in physics. In the work [1], V. Zakharov and A. Shabat showed that NLS equation can be applied in the study of optical self-focusing and splitting of optical beams. This equation belongs to the class of equations that can be solvable using the inverse scattering method for a Dirac-type operator. This was shown in the works of V.E. Zakharov and A.B. Shabat [1], L.A. Takhtadjan and L.D. Fadeev [2], M. Ablowitz, D. Kaup, A. Newell and H. Segur [3].

Several different form of solutions of the NLS equation have been studied by a large number of authors through their different method. The nonzero boundary soliton solutions for the NLS equation have been given by V.E.Zakharov and A.B. Shabat [5] in 1973. NSE was integrated in the class of "finite density" functions, i.e., functions for which $u(x,t) \to e^{i\alpha+2it}$, $u_x(x,t) \to 0$ as $x \to \infty$. The n-soliton solution of the NSE in the case of a finite density was found in [6].

The IST theory for the defocusing NLS equation with nonzero boundary conditions (NZBCs) was studied by B. Gino, F. Emily and B. Prinari [7] and the focusing case has been studied by G. Biondini, G. Kovacic [8], F. Demontis, B. Prinari, C. van der Mee, F. Vitale [9].

In [4], V.K. Melnikov obtained evolutions of scattering data with respect to t for a self-adjoint Dirac operator with a potential that is an NSE solution with a self-consistent source of integral type. However, we note that in the above works NSE was considered in the class of "rapidly decreasing" functions, i.e. conditions that vanish in a certain way as the coordinate tends to infinity.

Formulation of the problem

We consider the following system of equations

$$iu_t - 2u|u|^2 + u_{xx} = -2i\sum_{n=1}^{N}(f_{1,n}^* g_{2,n}^* + f_{2,n} g_{1,n}) \tag{1}$$

$$\frac{\partial f_{1,n}}{\partial x} - u^* f_{2,n} + i\xi_n f_{1,n} = \frac{\partial f_{2,n}}{\partial x} - u f_{1,n} - i\xi_n f_{2,n} = 0, \quad n = 1, 2, ..., N, \tag{2}$$

$$\frac{\partial g_{1,n}}{\partial x} - u g_{2,n} - i\xi_n g_{1,n} = \frac{\partial g_{2,n}}{\partial x} - u^* g_{1,n} + i\xi_n g_{2,n} = 0, \quad n = 1, 2, ..., N \tag{3}$$

under the initial condition

$$u(x,0) = u_0(x), \quad x \in R. \tag{4}$$

The initial function $u_0(x)$, $x \in R$ satisfies the following properties:

1)

$$\int_{-\infty}^{0}(1-x)\left|u_0(x) - \rho e^{i\alpha_-}\right|dx + \int_{0}^{\infty}(1+x)\left|u_0(x) - \rho e^{i\alpha_+}\right|dx < \infty, \tag{5}$$

where $\rho > 0$ and $0 \le \alpha_\pm < 2\pi$ are arbitrary constants.

2) The system of equations (2) with coefficient $u_0(x)$ possesses exactly N eigenvalues $\xi_1(0),\xi_2(0),...,\xi_N(0)$.

We assume that the solution $u(x,t)$ of the equation (1) is sufficiently smooth and tends to its limits sufficiently rapidly as $x\to\pm\infty$, i.e., for all $t\ge 0$ satisfies the condition

$$\int_{-\infty}^{0}(1-x)\left|u(x,t)-\rho e^{i\alpha_- -2i\rho^2 t}\right|dx+\int_{0}^{\infty}(1+x)\left|u(x,t)-\rho e^{i\alpha_+ -2i\rho^2 t}\right|dx+\int_{-\infty}^{\infty}\sum_{k=1}^{2}\left|\frac{\partial^k u(x,t)}{\partial x^k}\right|dx<\infty. \quad (6)$$

the functions $F_n(x,t)$ and $G_n(x,t)$ satisfies conditions

$$\int_{-\infty}^{\infty}G_n^T(s,t)F_n(s,t)ds=\omega_n(t),\ \ t\ge 0,\ n=1,2,...,N\,, \quad (7)$$

here $\omega_n(t)$, $n=1,2,...,N$ are given and the continuous functions of t.

Scattering problem with NZBCs

Consider the system of linear equations on the real line R

$$(L-\xi I)f=0, \quad (8)$$

where $f=f(x,\xi)$ is vector-column function and

$$L(t)=i\begin{pmatrix}\frac{\partial}{\partial x} & -\bar{u}(x,t)\\ u(x,t) & -\frac{\partial}{\partial x}\end{pmatrix},\ t\ge 0.$$

There we present some necessary facts for our further exposition from the theory of the direct and inverse scattering problem for the system of equations (8).

We define the Jost solutions of the system (8) with the following asymptotic values

$$\varphi\sim\begin{pmatrix}1\\ \frac{i(\xi-p)}{\rho}e^{i\alpha-2i\rho^2 t}\end{pmatrix}e^{-ipx},\ \text{as}\ \ x\to-\infty,$$
$$\bar{\varphi}\sim\begin{pmatrix}-\frac{i(\xi-p)}{\rho}e^{-i\alpha+2i\rho^2 t}\\ 1\end{pmatrix}e^{ipx},\ \text{as}\ \ x\to-\infty, \quad (9)$$

$$\psi \sim \begin{pmatrix} -\dfrac{i(\xi - p)}{\rho} e^{-i\beta + 2i\rho^2 t} \\ 1 \end{pmatrix} e^{ipx}, \text{ as } x \to -\infty,$$

$$\bar{\psi} \sim \begin{pmatrix} 1 \\ \dfrac{i(\xi - p)}{\rho} e^{i\beta - 2i\rho^2 t} \end{pmatrix} e^{-ipx}, \text{ as } x \to \infty,$$

where

$$p(\xi) = \sqrt{\xi^2 - \rho^2}, \tag{10}$$

here and below we will use the standard Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In order to discuss the analytic region of Jost functions, we need to discuss it on the proper complex plane. A two-sheeted Riemann surface is introduced to handle the branching of the eigenvalues, namely $p(\xi) = \sqrt{\xi^2 - \rho^2}$, the branch points can be easily derived by $\xi^2 - \rho^2 = 0$, i.e., $\xi = \rho$ and $\xi = -\rho$.

The variable p is then thought of as belonging to a Riemann surface Γ consisting of a sheet Γ_+ and a sheet Γ_- which both coincide with the complex plane cut along the semi lines

$$\Sigma = (-\infty, -\rho] \cup [\rho, \infty)$$

with its edges glued in such a way that $p(\xi)$ is continuous through the cut. The variable p is thought of as belonging to the complex plane consisting of the upper half complex plane Γ_+ and the lower half complex plane Γ_- glued together along the whole real line. For all $\xi \in \Sigma$, the branch of the square root is fixed by the condition $sign\, p(\xi) = sign\,\xi$.

It can be shown that

$$\frac{d}{dx}\det(\varphi, \bar{\varphi}) = 0 \text{ and } \frac{d}{dx}\det(\psi, \bar{\psi}) = 0. \tag{11}$$

From (9) and (11) it follows that

$$\det(\varphi, \bar{\varphi}) = \frac{2p(\xi - p)}{\rho^2},\ \det(\psi, \bar{\psi}) = \frac{2p(\xi - p)}{\rho^2}. \tag{12}$$

For real p and ξ pairs of vector functions $\{\varphi,\bar{\varphi}\}$ and $\{\psi,\bar{\psi}\}$ form a fundamental system of solutions to (8), so, there is a functions $a(t,\xi)$, $b(t,\xi)$ that for solutions $\{\varphi,\bar{\varphi}\}$ and $\{\psi,\bar{\psi}\}$

$$\varphi(x,t,\xi)=a(t,\xi)\bar{\psi}(x,t,\xi)+b(t,\xi)\psi(x,t,\xi), \text{ as } \xi \text{ из } R^1\setminus[-\rho,\rho]. \tag{13}$$

The coefficients $a(\xi,t)$ and $b(\xi,t)$ are called transition coefficients. From relations (11) and (12) we obtain

$$|a(\xi,t)|^2-|b(\xi,t)|^2=1, \tag{14}$$

where the functions $a(\xi)$ and $b(\xi)$ are independent of x and

$$a(\xi,t)=\frac{\rho^2}{2p(\xi-p)}\det(\varphi(x,\xi,t),\psi(x,\xi,t)), \tag{15}$$

$$b(\xi,t)=\frac{\rho^2}{2p(\xi-p)}\det(\varphi(x,\xi,t),\bar{\psi}(x,\xi,t)).$$

The Jost solutions $\psi(x,\xi,t)e^{-ipx}$ and $\varphi(x,\xi,t)e^{ipx}$ are analytic for $\xi\in\Gamma_+$, while the Jost solutions $\bar{\psi}(x,\xi,t)e^{ipx}$ and $\bar{\varphi}(x,\xi,t)e^{-ipx}$ are analytic for $\xi\in\Gamma_-$.

For each fixed x, there are asymptotes for $|\xi|\to\infty$

$$\varphi(x,\xi,t)e^{ipx}=\begin{pmatrix}1\\ \frac{i(\xi-p)}{\rho}e^{i\alpha-2ip^2t}\end{pmatrix}+\underline{\underline{O}}\left(\frac{|1+\xi-p|}{|\xi|}\right), \tag{16}$$

$$\psi(x,\xi,t)e^{-ipx}=\begin{pmatrix}-\frac{i(\xi-p)}{\rho}e^{-i\beta+2ip^2t}\\ 1\end{pmatrix}+\underline{\underline{O}}\left(\frac{|1+\xi-p|}{|\xi|}\right), \tag{17}$$

where ξ from Γ_+, and

$$\bar{\psi}(x,\xi,t)e^{ipx}=\begin{pmatrix}1\\ \frac{i(\xi-p)}{\rho}e^{i\beta-2ip^2t}\end{pmatrix}+\underline{\underline{O}}\left(\frac{|1+\xi-p|}{|\xi|}\right), \tag{18}$$

$$\bar{\varphi}(x,\xi,t)e^{-ipx}=\begin{pmatrix}-\frac{i(\xi-p)}{\rho}e^{-i\alpha+2ip^2t}\\ 1\end{pmatrix}+\underline{\underline{O}}\left(\frac{|1+\xi-p|}{|\xi|}\right), \tag{19}$$

where ξ from Γ_-.

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It follows from the analyticity properties of the Jost solutions and equality (15) that the function $a(\xi,t)$ can be analytically continued to the sheet Γ_+ excluding branch points $\xi=\pm\rho$.

From (16) and (17) we obtain that for $|\xi|\to\infty$, the function $a(\xi,t)$ has the asymptotics

$$a(\xi,t)=1+O\left(\frac{1}{|\xi|}\right) \text{ as } \operatorname{Im}\xi>0$$

and

$$a(\xi,t)=e^{-i\theta}+O\left(\frac{1}{|\xi|}\right) \text{ as } \operatorname{Im}\xi<0$$

where we recall $\theta=\alpha_+-\alpha_-$.

Similarly, the function $\bar{a}(\xi,t)$ can be analytically continued to the sheet Γ_-, excluding branch points $\xi=\pm\rho$.

From (16) and (18) it follows that when $|\xi|\to\infty$, $\xi\in\Sigma$

$$b(\xi,t)=O\left(\frac{1}{|\xi|}\right).$$

It is well known that the scattering problem admits two involutions: $P:(\xi,p)\mapsto(\xi^*,p^*)$ and $J:(\xi,p)\mapsto(\xi,-p)$. For involution $P(\xi)$ we obtain the corresponding symmetry relations between Jost solutions

$$\sigma_1\bar{\psi}^*(x,\xi,t)=\psi(x,P(\xi),t), \text{ for } \xi \text{ from } \Gamma_-,$$

$$\sigma_1\varphi^*(x,\xi,t)=\bar{\varphi}(x,P(\xi),t), \text{ for } \xi \text{ from } \Gamma_+.$$

For any ξ from Γ_+ and Γ_- respectively, we have

$$\bar{\psi}^*(x,J(\xi),t)=\frac{\xi+p}{i\rho}\sigma_1\bar{\psi}(x,\xi,t)e^{-i\alpha_++2i\rho^2t},$$

$$\varphi^*(x,J(\xi),t)=\frac{\xi+p}{i\rho}\sigma_1\varphi(x,\xi,t)e^{-i\alpha_-+2i\rho^2t},$$

$$\psi^*(x,J(\xi),t)=-\frac{\xi+p}{i\rho}\sigma_1\psi(x,\xi,t)e^{i\alpha_+-2i\rho^2t},$$

$$\bar{\varphi}^*(x,J(\xi),t)=-\frac{\xi+p}{i\rho}\sigma_1\bar{\varphi}(x,\xi,t)e^{i\alpha_--2\rho^2t}.$$

Correspondingly, from (16)-(19), we can obtain the symmetries of the scattering coefficients:

$$a(\xi,t)=\bar{a}(J(\xi),t).$$

It follows from the analyticity with respect to the function $a(\xi,t)$ on Γ_+ and from the asymptotics (11), (12) that the function $a(\xi,t)$ can have only a finite number of zeros on the sheet Γ_+. These zeros will be denoted by $\xi_1,\xi_2,...,\xi_N$. In [2] it is shown that all zeros are simple and all belong to the $(-\rho,\rho)$.

It follows from representation (15) that if $a(\xi_n,t)=0$, then the columns $\varphi(x,\xi,t)$ and $\psi(x,\xi,t)$ are linearly dependent at $\xi=\xi_n$, i.e.,

$$\varphi(x,\xi_n,t)=c_n(t)\psi(x,\xi_n,t), \quad n=1,2,...,N. \tag{20}$$

Note that the vector-functions

$$h_n(x,t)=\frac{\left.\frac{d}{d\xi}(\varphi-c_n\psi)\right|_{\xi=\xi_n}}{\dot{a}(\xi_n,t)}, \quad n=1,2,...,N.$$

The following integral representations hold for the Jost solutions

$$\psi(x,\xi,t)=\begin{pmatrix}-\frac{i(\xi-p)}{\rho}e^{-i\beta+2i\rho^2t}\\ 1\end{pmatrix}e^{ipx}+\int_{-\infty}^{x}K(x,y,t)\begin{pmatrix}-\frac{i(\xi-p)}{\rho}e^{-i\beta+2i\rho^2t}\\ 1\end{pmatrix}dy, \tag{21}$$

where

$$K^{\pm}(x,y,t)=\begin{pmatrix}K_{11}^{\pm}(x,y,t) & K_{12}^{\pm}(x,y,t)\\ K_{21}^{\pm}(x,y,t) & K_{22}^{\pm}(x,y,t)\end{pmatrix}.$$

In representation (21) the kernel $K(x,y,t)$ does not dependent on ξ and the related to the potential $u(x,t)$ as the following:

$$2K_{21}(x,x,t)=\rho e^{i\beta-2i\rho^2t}-u(x,t), \tag{22}$$

It is well known that the components of the kernel $K^+(x,y,t)$ for $y>x$ are solutions of the system of Gelfand-Levitan-Marchenko integral equations:

$$K^+(x,y)+F(x+y)+\int_x^{\infty}K^+(x,s)F(s+y)ds=0, \quad y\geq x \tag{23}$$

where

$$F(x)=\begin{pmatrix} F_1(x) & F_2^*(x) \\ F_2(x) & F_1(x) \end{pmatrix}, \quad e(x,z)=e^{\frac{i}{2}\left(z-\frac{\rho^2}{z}\right)x}$$

$$\begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix}=\frac{1}{4\pi}\int_{-\infty}^{\infty} r(z,t)e(x,z)\begin{pmatrix} \dfrac{-i\rho e^{i\beta-2i\rho^2 t}}{z} \\ 1 \end{pmatrix}dz-\frac{1}{2}\sum_{n=1}^{N}\frac{c_n(t)}{\dot{a}(z_n,t)z_n}\cdot\begin{pmatrix} \rho e^{-i\beta+2i\rho^2 t} \\ iz_n \end{pmatrix}e(x,z_n).$$

Definition. The set of the quantities $\{a(\xi,t), b(\xi,t), \xi_n(t), c_n(t), n=1,2,...,N\}$ is called the scattering data for equation (8).

Time evolution

If the potential $u(x,t)$ in the system of equations (8) depends on t, then its solution f must also depend on t. Let this time dependence have the form

$$\frac{\partial}{\partial t}f=A(x,t,\xi)f, \tag{24}$$

where

$$A=\begin{pmatrix} i|u|^2+2i\xi^2 & -iu_x^*-2\xi u^* \\ iu_x-2\xi u & -i|u|^2-2i\xi^2 \end{pmatrix}.$$

The compatibility condition for linear systems (7) and (21) is

$$\frac{\partial L}{\partial t}+[L,A]=0, \tag{25}$$

where $[L,A]=LA-AL$.

Let the function $u(x,t)$ be a solution of equation (1), from the class of functions (5). Consider equation (8) with a coefficient $u(x,t)$.

Let $f(x,\xi,t)$ be solution of the equation (8) and let $\phi_n(x,\xi,t)$, $n=1,2,...,2N$ be any functions, which satisfy the conditions

$$\frac{\partial \phi_n}{\partial x}=G_n^T f, \quad n=1, 2, ..., 2N.$$

Then, the function $G_n(x,t)$ satisfy the equalities

$$G_n^T\sigma_3 f+i(\xi-\xi_n)\phi_n=0, \; n=1, 2, ..., 2N$$

and the function

$$H=\frac{\partial f}{\partial t}-Af+\sum_{n=1}^{2N}F_n\phi_n$$

satisfies the equation (8) for any $\xi \in \Sigma$.

Let $\varphi(x,\xi,t)$ be a Jost solution of the equation $L(t)\varphi = \xi\varphi$. By differentiating this relation with respect to t, we obtain the equation

$$\frac{\partial L}{\partial t}\varphi + L\frac{\partial \varphi}{\partial t} = \xi\frac{\partial \varphi}{\partial t}. \tag{26}$$

By substituting $\frac{\partial L}{\partial t}$ (25) into (26), we obtain the equation

$$(L-\xi)\left(\frac{\partial \varphi}{\partial t} - A\varphi\right) = -iG\varphi, \tag{27}$$

whose solution we seek in the form

$$\frac{\partial \varphi}{\partial t} - A\varphi = \alpha(x,t)\psi + \beta(x,t)\varphi. \tag{28}$$

Where $G = \begin{pmatrix} 0 & \bar{g} \\ g & 0 \end{pmatrix}$, $g(x,t) = -2i\sum_{n=1}^{N}(\phi_{1,n}^{*}\psi_{2,n}^{*} + \phi_{2,n}\psi_{1,n})$. For the functions $\alpha(x,t)$ and $\beta(x,t)$, we obtain the equation

$$\sigma_3\frac{\partial \alpha}{\partial x}\psi + \sigma_3\frac{\partial \beta}{\partial x}\varphi = -G\varphi. \tag{29}$$

By multiplying Eq. (29) by $\sigma_1\varphi$ and $\sigma_1\psi$, we obtain

$$\frac{\partial \alpha}{\partial x} = \frac{i\rho^2}{2p(\xi - p)}\frac{\sigma_1\varphi G\varphi}{a}, \quad \frac{\partial \beta}{\partial x} = \frac{i\rho^2}{2p(\xi - p)}\frac{\sigma_1\psi G\psi}{a}. \tag{30}$$

Relation (8) implies that $\frac{\partial \varphi}{\partial t} - A\varphi \to (2i\xi p + i\rho^2)\varphi$ as $x \to -\infty$ therefore, from (24) we have $a(x,t) \to 0$, $\beta(x,t) \to 2i\xi p + i\rho^2$ as $x \to -\infty$. By solving (30), we obtain

$$\alpha(x,t) = \frac{1}{a}\int_{-\infty}^{x}\sigma_1\varphi G\varphi ds, \quad \beta(x,t) = -\frac{1}{a}\int_{-\infty}^{x}\sigma_1\psi G\psi ds - 2i\xi p - i\rho^2.$$

Therefore, relation (13) can be represented in the form

$$\frac{\partial \varphi}{\partial t} - A\varphi = \frac{1}{a}\int_{-\infty}^{x}\sigma_1\varphi G\varphi ds \cdot \psi - \left(\frac{1}{a}\int_{-\infty}^{x}\sigma_1\psi G\psi ds + 2i\xi p + i\rho^2\right)\cdot\varphi \tag{28}$$

By using (13) and by passing to the limit as $x \to \infty$ in (28), we obtain

$$\dot{a} = -\frac{i\rho^2}{2p(\xi - p)} \int_{-\infty}^{\infty} \sigma_1 \psi G \varphi ds,$$

$$\dot{b} = -(2i\xi p + i\rho^2)b + \frac{i\rho^2}{2p(\xi - p)a} \int_{-\infty}^{\infty} \sigma_1 \varphi G \varphi ds - \frac{i\rho^2 b}{2p(\xi - p)a} \int_{-\infty}^{\infty} \sigma_1 \psi G \varphi ds .$$

As in the continuous spectrum, one can show that

$$\dot{c}_n = -(4i\xi_n p_n + 2i\rho^2)c_n - \frac{i\rho^2}{2p_n(\xi_n - p_n)} \int_{-\infty}^{\infty} \sigma_1 h_n R \varphi_n ds ,$$

$$\frac{d\xi_n}{dt} = \frac{\int_{-\infty}^{\infty} \sigma_1 \varphi G \varphi dx}{2\int_{-\infty}^{\infty} \varphi_{n1}\varphi_{n2} dx}, \quad n = 1, 2, 3, \ldots, N .$$

Theorem 1. If the function $u(x,t)$ is a solution of the equation (1) in the class of functions (6), then the scattering data of the system (8) with the function $u(x,t)$ depend on t as follows:

$$\dot{a} = -\frac{i\rho^2}{2p(\xi - p)} \int_{-\infty}^{\infty} \sigma_1 \varphi G \varphi ds,$$

$$\dot{b} = (2i\xi p + i\rho^2)b + \frac{i\rho^2}{2p(\xi - p)a} \int_{-\infty}^{\infty} \sigma_1 \varphi G \varphi ds - \frac{i\rho^2}{2p(\xi - p)a} \int_{-\infty}^{\infty} \sigma_1 \psi G \psi ds ,$$

$$\dot{c}_n = -(4i\xi_n p_n + 2i\rho^2)c_n - \frac{i\rho^2}{2p_n(\xi_n - p_n)} \int_{-\infty}^{\infty} \sigma_1 h_n R \varphi_n ds ,$$

$$\frac{d\xi_n}{dt} = \frac{\int_{-\infty}^{\infty} \sigma_1 \varphi_n G \varphi_n dx}{2\int_{-\infty}^{\infty} \varphi_{n1}\varphi_{n2} dx}, \quad n = 1, 2, 3, \ldots, N .$$

The obtained relations determine completely the evolution of the scattering data for the system (8), which allow as to find the solution of the problem (1)-(3) by using the inverse scattering problem method.

Corollary. If we get $g(x,t) = -2i\sum_{n=1}^{N}(\phi_{1,n}^{*}\psi_{2,n}^{*} + \phi_{2,n}\psi_{1,n})$ then

$$\int_{-\infty}^{\infty}\sigma_1\varphi G\varphi dx=0,\quad \int_{-\infty}^{\infty}\sigma_1\psi G\varphi dx=0,$$

$$\int_{-\infty}^{\infty}\sigma_1 h_n G\varphi_n dx=\frac{2p_n(\xi_n-p_n)}{\rho^2}(\omega_n(t)-\omega_n^*(t)),\quad \int_{-\infty}^{\infty}\sigma_1\varphi_n G\varphi_n dx=0.$$

In this case

$$\frac{d\xi_n}{dt}=0,\ \dot{c}_n=-(4i\xi_n p_n+2i\rho^2-i(\omega_n(t)-\omega_n^*(t))c_n.$$

Example. Let

$$u_0=\sqrt{2}e^{-4it}\frac{ie^{-x}+e^x}{e^{-x}+e^x}.$$

In this case, the scattering data system of equations (8) with potential u_0 has

$$a(t,\xi)=\frac{\xi-p-1-i}{\xi+p-1+i},\ b(t,\xi)=0,\ c_1=\frac{i(1+i)}{\sqrt{2}},\ \xi=1,\ p=i$$

Using results theorem 1, we can find

$$\frac{d\xi_1}{dt}=0,\ c_1(t)=\frac{i-1}{\sqrt{2}}e^{4t-4it}\cdot\exp\int_0^t i(\omega_n(\tau)-\omega_n^*(\tau))d\tau.$$

Solving the inverse problem we get

$$u(x,t)=\sqrt{2}e^{-4it}\frac{e^x+ie^{-x+2g(t)}}{e^x+e^{-x+2g(t)}},$$

$$\varphi_1=\frac{\alpha_1}{e^{-x}+e^x},\quad \varphi_2=\alpha_1\frac{-(1-i)}{\sqrt{2}}e^{-4it}\cdot\frac{1}{e^{-x}+e^x},$$

where $g(t)=i\int_0^t(\omega_n(\tau)-\omega_n^*(\tau))d\tau$.

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