SHTOLS TEOREMASI VA UNING TATBIQLARI

Urganch davlat universiteti Fizika-matematika fakulteti matematika yo'nalishi 1-kurs magistranti **Jumaniyozova Xayriniso Xamidbek qizi**

Annotatsiya: Shtols teoremasi matematik analizdagi muhim teorema hisoblanadi. Quyida teorema isboti va xalqaro olimpiadalarda tadbiqlarini ko'rib chiqamiz.

Kalit sozlar: limit, Shtols teoremasi, monoton ketma-ketliklar.

Teorema(Shtols). Bizga ikkita $(a_n)_{n\geq 1}$ va $(b_n)_{n\geq 1}$ ketma-ketliklar berilgan bo'lsin:

1) $(b_n)_{n\geq 1}$ ketma-ketlik qat'iy o'suvchi va

$$\lim_{n\to\infty} b_n = \infty$$

bo'lsin;

2) Quyidagi limit mavjud bo'lsin:

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = l$$

U holda quyidagi ketma-ketlik yaqinlashuvchi va

$$\left\{\frac{a_n}{b_n}\right\}$$

uning limiti l ga teng, ya'ni

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l$$

Isbot. Teorema shartida $(b_n)_{n\geq 1}$ ketma-ketlik qat'iy o'suvchi va limiti ∞ ga teng. Demak, ketma-ketligimiz biror joydan ($n = n_0$ chi hadidan boshlab) musbat qiymat qabul qilib boshlaydi va

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = l$$

limiti mavjud.

$\forall \varepsilon > 0$ berilganda ham $\exists m \in N$ mavjudki $\forall n \geq m$ natural sonlar uchun

$$\left|\frac{a_{n+1}-a_n}{b_{n+1}-b_n}-l\right|<\varepsilon$$

bo'ladi. Bundan quyidagini yozib olamiz

$$(l-\varepsilon)\left(b_{n+1}-b_n\right)<a_{n+1}-a_n<(l+\varepsilon)\left(b_{n+1}-b_n\right)$$

endi n ni k bilan, $k+1$ bilan $k+2$ bilan va hokazo $n-1$ bilan almashtirib yozamiz:

$$(l-\varepsilon)\left(b_{k+1}-b_k\right)<a_{k+1}-a_k<(l+\varepsilon)\left(b_{k+1}-b_k\right)$$

$$(l-\varepsilon)\left(b_{k+2}-b_{k+1}\right)<a_{k+2}-a_{k+1}<(l+\varepsilon)\left(b_{k+2}-b_{k+1}\right)$$

$$(l-\varepsilon)\left(b_{k+3}-b_{k+2}\right)<a_{k+3}-a_{k+2}<(l+\varepsilon)\left(b_{k+3}-b_{k+2}\right)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(l-\varepsilon)\left(b_n-b_{n-1}\right)<a_n-a_{n-1}<(l+\varepsilon)\left(b_n-b_{n-1}\right)$$

Bu ifodalarni qo'shib yuborsak

$$(l-\varepsilon)\left(b_n-b_k\right)<a_n-a_k<(l+\varepsilon)\left(b_n-b_k\right)$$

bu ifodani b_n ga bo'lamiz:

$$(l-\varepsilon)\left(1-\frac{b_k}{b_n}\right)<\frac{a_n}{b_n}-\frac{a_k}{b_n}<(l+\varepsilon)\left(1-\frac{b_k}{b_n}\right)$$

$$l-\varepsilon+\frac{a_k+(\varepsilon-l)b_k}{b_n}<\frac{a_n}{b_n}<l+\varepsilon+\frac{a_k-(\varepsilon+l)b_k}{b_n}$$

bilamizki

$$\lim_{n\to\infty}\frac{a_k+(\varepsilon-l)b_k}{b_n}=\lim_{n\to\infty}\frac{a_k-(\varepsilon+l)b_k}{b_n}=0$$

yuqoridagi $\forall\varepsilon>0$ kora $\exists k\in N$ mavjudki $\forall n\geq k$ natural sonlar uchun quyidagilar o'rinli:

$$-\varepsilon<\frac{a_k+(\varepsilon-l)b_k}{b_n}<\varepsilon$$

$$-\varepsilon<\frac{a_k-(\varepsilon+l)b_k}{b_n}<\varepsilon.$$

Endi $p=\max\{m,p\}$ deb olsak, $\forall n\geq p$ natural sonlar uchun yuqoridagi ikkita tengsizligimiz bir vaqtda bajariladi. Quyidagiga ega bo'lamiz

$$l-2\varepsilon<l-\varepsilon+\frac{a_k+(\varepsilon-l)b_k}{b_n}<\frac{a_n}{b_n}<l+\varepsilon+\frac{a_k-(\varepsilon+l)b_k}{b_n}<l+2\varepsilon$$

$$l-2\varepsilon<\frac{a_n}{b_n}<l+2\varepsilon$$

$$\left|\frac{a_n}{b_n}-l\right|<2\varepsilon$$

ε sonining ixtiyoriyligidan 2ε ixtiyoriy son bo'ladi. Ketma-ketlik limit tarifidan

$$\left\{\frac{a_n}{b_n}\right\}$$

Ketma-ketlik yaqinlashuvchi va limiti l ga teng.
Teorema to'liq isbotlandi.
Endi teoremani olimpiada misoliga tatbiq etamiz

Misol. Quyidagi ketma-ketlikning $n\to\infty$ limitni hisoblang:

$$z_n=\left(1+11^n\right)^{\frac{1}{n+2}}$$

Yechilishi: $z_n=\left(1+11^n\right)^{\frac{1}{n+2}}$ bu ketma-ketlikni Shtols teoremasini qanoatlantirishi uchun quyidagicha shakl almashtirish bajaramiz:

$$z_n=\left(1+11^n\right)^{\frac{1}{n+2}}=e^{\frac{1}{n+2}\ln\left(1+11^n\right)}.$$

Endi z_n ketma-ketlikni x_n va y_n ketma-ketliklar orqali ifodalaymiz: $x_n=\ln\left(1+11^n\right)$, $y_n=n+2$. Bunda ketma-ketlik quyidagi ko'rinishni oladi:

$$z_n=e^{\frac{x_n}{y_n}}.$$

Bilamizki,

$$\lim_{n\to\infty}z_n=\lim_{n\to\infty}e^{\frac{x_n}{y_n}}=e^{\lim\limits_{n\to\infty}\frac{x_n}{y_n}}$$

munosabat o'rinli. Endi $\lim\limits_{n\to\infty}\frac{x_n}{y_n}$ qiymatini topish bilan shug'ullanamiz. Bu yerda $x_n=\ln\left(1+11^n\right)$, $y_n=n+2$ hamda y_n qat'iy o'suvchi: $y_{n+1}>y_n$. Shtols teoremasining shartlarini qanoatlantirdi. Endi

$$\lim_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n}$$

limitni hisoblaymiz:

$$\lim_{n\to\infty}\frac{\ln\left(1+11^{n+1}\right)-\ln\left(1+11^{n}\right)}{(n+3)-(n+2)}=\lim_{n\to\infty}\ln\left(\frac{1+11^{n+1}}{1+11^{n}}\right)=\lim_{n\to\infty}\ln\left(\frac{\frac{1}{11^{n}}+11}{\frac{1}{11^{n}}+1}\right)=\ln 11$$

Shtols teoremasiga ko'ra

$$\lim_{n\to\infty}\frac{x_n}{y_n}=\lim_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n}$$

munosabat o'rinli. Bundan $\lim_{n\to\infty}\frac{x_n}{y_n}=\ln 11$ ekanligi kelib chiqadi. Endi bu qiymatni o'rniga qo'yib berilgan ketma-ketlikning $n\to\infty$ dagi limitini hisoblaymiz:

$$\lim_{n\to\infty} z_n=\lim_{n\to\infty} e^{\frac{x_n}{y_n}}=e^{\lim_{n\to\infty}\frac{x_n}{y_n}}=e^{\ln 11}=11$$

Demak,

$$\lim_{n\to\infty}\left(1+11^{n}\right)^{\frac{1}{n+2}}=11.$$

Foydalanilgan adabiyotlar:

1. **Azlarov T., Mansurov H** Matematik analiz: Unversitet va ped. Institut talabalari uchun darslik: 2 qismli. 1-q. –qayta ishlangan va toldirilgan 2-nashri. –T.: Oqituvchi, 1994.-416 b
2. Brainkart.com.