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Direxle integrali va Lobachevskiy formulasi

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Annotatsiya. Bugungi kunda respublikamizda ta'lim sohasida olib borilayotgan islohotlar talabalar uchun zamon talabiga javob beradigan dars jarayoni, uslibiy ko'rsatmalar, uslubiy qo'llanmalar yaratishni taqozo qiladi,

Biz bu mavzuda Dirixle integralini hisoblash metodini o'rganamiz.

1-ta'rif. Quyidagi

$$D(a)=\int_0^{+\infty}\frac{\sin ax}{x}dx$$

parametrga bog'liq xosmas integralga Dirixle integrali deyiladi.

Endi Dirixle integralini hisoblashni ko'ramiz. Quyidagi

$$F(a,b)=\int_0^{+\infty}e^{-bx}\frac{\sin ax}{x}dx,\quad b\geq 0 \qquad (1)$$

parametrga bog'liq xosmas integralni olamiz.

$$f(x,b)=\begin{cases}e^{-bx}\dfrac{\sin ax}{x}, x\neq 0\\ a, x=0\end{cases},\quad b\geq 0$$

funksiya $b\geq 0$, $0\leq x\leq +\infty$ da uzluksiz, chunki

$$\lim_{x\to 0+0}e^{-bx}\frac{\sin ax}{x}=a=f(0)$$

ya'ni, funksiya $x=0$ nuqtada uzluksiz va $0<x<+\infty$ da e^{-bx} va $\dfrac{\sin ax}{x}$

funksiyalar uzluksizligidan $e^{-bx}\dfrac{\sin ax}{x}$ funksiyaning uzluksizligi kelib chiqadi.

Diexle alomatidan foydalanib

$$\int_0^{+\infty}\frac{\sin ax}{x}dx$$

xosmas integralni tekis yaqinlashishini ko'rsatamiz.

1) $\forall t\geq 0$ da

$$\left|\int_0^t \sin axdx\right| = \left|\frac{\cos at}{a} - \frac{1}{a}\right| \le \frac{2}{a}$$

2) har bir tayin $a>0$ da $\frac{1}{x}$ funksiya 0 funksiyaga tekis yaqinlashadi.

Direxle alomatiga ko'ra yaqinlashuvchi bo'ladi.

$b \ge 0$ va $0 \le x < \infty$ da e^{-bx} funksiya monoton va chegaralangan.

$$\left|e^{-bx}\right| \le 1.$$

Shuningdek,

$$\int_0^{+\infty} \frac{\sin ax}{x} dx$$

xosmas integral tekis yaqinlashuvchi. U xolda, Abel alomatiga ko'ra (1) parametrga bog'liq xosmas integral tekis yaqinlashuvchi bo'ladi.

$f(x,b)$ funksiya $M = \{(x,b) \in \mathbb{R}^2; x \in [0,+\infty), b > 0\}$ to'plamda berilgan va uzluksiz. Shuningdek, $f_a'(x,a) = e^{-bx}\cos ax$ xususiy hosila mavjud va u M to'plamda berilgan va uzluksiz.

$$\int_0^{+\infty} f_a'(x,a)dx = \int_0^{+\infty} e^{-bx}\cos axdx$$

$$\left|e^{-bx}\cos ax\right| \le e^{-bx} \quad \text{va} \quad \int_0^{+\infty} e^{-bx}dx = \frac{e^{-bx}}{-b}\Bigg|_0^{+\infty} = \frac{1}{b}$$

Veyereshtrass alomatiga ko'ra

$$\int_0^{+\infty} f_a'(x,a)dx$$

tekis yaqinlashuvchi.

Endi quyidagi teoremadan foydalanamiz.

Faraz qilaylik, $f(x,y)$ funksiya $M_0 = \{(x,y) \in \mathbb{R}^2; x \in [a,+\infty), y \in [c,d]\}$ to'plamda berilgan bo'lsin.

1-teorema. $f(x,y)$ funksiya quyidagi shartlarni qanoatlantirsin.

1) $f(x,y)$ funksiya M_0 to'lamda uzluksiz;

2) $f_y'(x,y)$ xususiy hosila mavjud va u M_0 to'plamda uzluksiz;

3) Har bir tayin $y \in [c,d]$ da

$$F(y) = \int_a^{+\infty} f(x,y)\,dx$$

integral yaqinlashuvchi;

4) Ushbu $\int_a^{+\infty} f_y'(x,y)dx$ integral $[c,d]$ da tekis yaqinlashuvchi.

U holda $F(y)$ funksiya $[c,d]$ da $F'(y)$ hosilaga ega va

$$F'(y) = \int_a^{+\infty} f_y'(x,y)\,dx$$

bo‘ladi.

Bu teoremani isbotini foydalanilgan adabiyotlardan ko‘rib olishingiz mumkin.

Bizning

$$f(x,b) = \begin{cases} e^{-bx}\dfrac{\sin ax}{x}, x \neq 0 \\ a, x = 0 \end{cases}, \ b > 0$$

funksiya M to‘plamda yuqoridagi teoremaning barcha shartlarini qanoatlantiradi.

U holda

$$F_a(a,b) = \int_0^{+\infty} e^{-bx}\cos ax dx$$

tenglik o‘rinli bo‘ladi. Bu integralni ikki marta bo‘laklashdan foydalanib

$$F_a(a,b) = \int_0^{+\infty} e^{-bx}\cos ax dx = \frac{b}{b^2 + a^2}$$

tenglik hosil bo‘lamiz. U holda

$$F(a,b) = \int \frac{b}{b^2 + a^2}da = arctg\frac{a}{b} + \varphi(b)$$

bo‘ladi.

$a = 0$ da $F(0,b) = 0 + \varphi(b) = 0$ bo‘ladi. Bundan $\varphi(b) = 0$ kelib chiqadi.

U holda

$$F(a,b) = arctg\frac{a}{b}$$

bo‘ladi.

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$$D(a)=\int_0^{+\infty}\frac{\sin ax}{x}dx=\lim_{b\to 0+0}\int_0^{+\infty}e^{-bx}\frac{\sin ax}{x}dx=\lim_{b\to 0+0}F(a,b)=$$

$$=\lim_{b\to 0+0}arctg\frac{a}{b}=\frac{\pi}{2}sign(a)$$

bo'ladi. Demak,

$$D(a)=\int_0^{+\infty}\frac{\sin ax}{x}dx=\frac{\pi}{2}sign(a).$$

Natija. $a=1$ da

$$\int_0^{+\infty}\frac{\sin x}{x}dx=\frac{\pi}{2},$$

$\frac{\sin x}{x}$ funksiya juft bo'lgani uchun, bu integralni quyidagicha ham yozish mumkin

$$\int_{-\infty}^{+\infty}\frac{\sin x}{x}dx=\pi.$$

Biz yuqorida Direxli integralini hisoblashning umumiy holini ko'rgan edik. Biz Lobachevskiy formulasini isbotlashda kerak bo'ladigan bazi lemmalarning isbotini ko'rib chiqamiz.

1-Lemma. $a\neq \square\,\pi$ uchun,

$$\frac{1}{\sin a}=\frac{1}{a}+\sum_{m=1}^{\infty}(-1)^m\left(\frac{1}{a-m\pi}+\frac{1}{a+m\pi}\right)$$

tenglik o'rinli.

Isbot. Bu tenglikni isbotlashda Furye qatoridan foydalanamiz.

$f(x)$ funksiya $[-\pi,\pi]$ da berilgan va integrallanuvchi bo'lsin.

Furye qatoti:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

bu yerda,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx,\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx,\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx,\quad \forall n\in\square$$

lar Furye koeffisiyentlari deyiladi.

$f(x)=\cos ax$ ni $[-\pi,\pi]$ segementda Furye qatoriga yoyamiz.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos axdx = \frac{2\sin a\pi}{a\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos ax\cos nxdx = (-1)^n \frac{\sin a\pi}{\pi}\left(\frac{1}{a-n}+\frac{1}{a+n}\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos ax\sin nxdx = 0$$

tengliklar o'rinli bo'ladi. U holda,

$$f(x) = \cos ax = \frac{\sin a\pi}{a\pi} + \sum_{n=1}^{\infty}(-1)^n \frac{\sin a\pi}{\pi}\left(\frac{1}{a-n}+\frac{1}{a+n}\right)\cos nx$$

tenglikni olamiz , bu tenglikda $x = 0$ desak,

$$1 = \frac{\sin a\pi}{a\pi} + \frac{\sin a\pi}{\pi}\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{a-n}+\frac{1}{a+n}\right)$$

$a \neq \square$ deb, tengflikning ikkala tomonini $\sin a\pi$ ga bo'lsak va $a\pi = b$ deb belgilasak, biz izlayotgan

$$\frac{1}{\sin b} = \frac{1}{b} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{b-n\pi}+\frac{1}{b+n\pi}\right)$$

tenglik hosil bo'ladi, bunda $b \neq \square\,\pi$. Lemma isbotlandi.

2-Lemma. $a \neq \square\,\pi$ uchun quyidagi tenglik o'rinli

$$\frac{1}{\sin^2 a} = \frac{1}{a^2} + \sum_{n=1}^{\infty}\left(\frac{1}{(a-n\pi)^2}+\frac{1}{(a+n\pi)^2}\right).$$

Isbot. Quyidagi formuladan hosila olishdan xosil bo'ladi.

$$ctga = \frac{1}{a} + \sum_{n=1}^{\infty}\frac{2a}{a^2-n^2\pi^2}.$$

$f(x)$ funksiya $0 \le x < \infty$ da uzluksiz va π -davri bo'lsin, $f(x+\pi) = f(x)$ va $f(\pi - x) = f(x)$ o'rinli bo'lsin $0 \le x < \infty$ sohada.

1-Teorema. Agar $f(x)$ funksiya yuqoridagi shartlarni qanoatlantirsa. Quyidagi Lobachevskiy formulasi o'rinli.

$$\int_0^\infty \frac{\sin^2 x}{x^2} f(x)dx = \int_0^\infty \frac{\sin x}{x} f(x)dx = \int_0^{\frac{\pi}{2}} f(x)dx .$$

Isbot. Quyidagi integralni olaylik

$$I = \int_0^\infty \frac{\sin x}{x} f(x)$$

biz bu integralni quyidagicha yozamiz.

$$I = \int_0^\infty \frac{\sin x}{x} f(x) = \sum_0^\infty \int_{v\frac{\pi}{2}}^{(v+1)\frac{\pi}{2}} \frac{\sin x}{x} f(x)dx$$

agar $v = 2m-1$ bo'lsa, $x = m\pi - t$ o'zgartirish kiritamiz, agar $v = 2m$ bo'lsa $x = m\pi + t$ o'zgartirish kiritamiz.

$$\int_{2m\frac{\pi}{2}}^{(2m+1)\frac{\pi}{2}} \frac{\sin x}{x} f(x) = (-1)^m \int_0^{\frac{\pi}{2}} \frac{\sin t}{m\pi + t} f(t)dt$$

$$\int_{(2m-1)\frac{\pi}{2}}^{2m\frac{\pi}{2}} \frac{\sin x}{x} f(x) = (-1)^{m-1} \int_0^{\frac{\pi}{2}} \frac{\sin t}{m\pi - t} f(t)dt$$

tengliklar o'rinli bo'ladi. U holda

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin t}{t} f(t)dt + \sum_{n=1}^\infty \int_0^{\frac{\pi}{2}} (-1)^m f(t)\left(\frac{1}{t-m\pi} - \frac{1}{t+m\pi}\right)\sin t dx$$

bu integralni soddalashtirsak quyidagi ko'rinishga keladi,

$$I = \int_0^{\frac{\pi}{2}} \sin t\left(\frac{1}{t} + \sum_{m=1}^\infty (-1)^m \left(\frac{1}{t-m\pi} + \frac{1}{t+m\pi}\right)\right) f(t)dt$$

1-lemmadan foydalansak

$$I = \int_0^{\frac{\pi}{2}} f(x)dx$$

tenglik hosil bo'ladi.Demak,

$$\int_0^\infty \frac{\sin x}{x} f(x)dx = \int_0^{\frac{\pi}{2}} f(x)dx$$

bo'ladi. Tenglikning bir tarafini isbotladik. Endi

$$\int_0^\infty \frac{\sin^2 x}{x^2} f(x)dx = \int_0^{\frac{\pi}{2}} f(x)dx$$

ni isbotlaymiz.

$$J=\int_0^{\infty}\frac{\sin^2 x}{x^2}f(x)dx$$

deb belgilab,quyidagi tenglikni olamiz

$$J=\sum_0^{\infty}\int_{v\frac{\pi}{2}}^{(v+1)\frac{\pi}{2}}\frac{\sin^2 x}{x^2}f(x)dx$$

agar $v=2m-1$ bo'lsa, $x=m\pi-t$ o'zgartirish kiritamiz, agar $v=2m$ bo'lsa $x=m\pi+t$ o'zgartirish kiritamiz.

$$\int_{(2m-1)\frac{\pi}{2}}^{2m\frac{\pi}{2}}\frac{\sin^2 x}{x^2}f(x)dx=\int_0^{\frac{\pi}{2}}\frac{\sin^2 t}{(m\pi-t)^2}f(t)dt$$

$$\int_{2m\frac{\pi}{2}}^{(2m+1)\frac{\pi}{2}}\frac{\sin^2 x}{x^2}f(x)dx=\int_0^{\frac{\pi}{2}}\frac{\sin^2 t}{(m\pi+t)^2}f(t)dt$$

tengliklar o'rinli. Bundan

$$J=\int_0^{\frac{\pi}{2}}\frac{\sin^2 t}{t^2}f(t)dt+\sum_{m=1}^{\infty}\int_0^{\frac{\pi}{2}}f(t)\left(\frac{1}{(t+m\pi)^2}+\frac{1}{(t-m\pi)^2}\right)\sin^2 tdt$$

kelib chiqadi. Bu integralni soddalashtirsak

$$J=\int_0^{\frac{\pi}{2}}\sin^2 t\left(\frac{1}{t^2}+\sum_{n=1}^{\infty}\left(\frac{1}{(t-m\pi)^2}+\frac{1}{(t+m\pi)^2}\right)\right)f(t)dt$$

hosil bo'ladi 2-lemmadan foydalansak

$$J=\int_0^{\frac{\pi}{2}}f(t)dt$$

bo'ladi. Demak,

$$J=\int_0^{\infty}\frac{\sin^2 x}{x^2}f(x)dx=\int_0^{\frac{\pi}{2}}f(x)dx$$

tenglik hosil boladi. U holda bizda

$$\int_0^{\infty}\frac{\sin^2 x}{x^2}f(x)dx=\int_0^{\infty}\frac{\sin x}{x}f(x)dx=\int_0^{\frac{\pi}{2}}f(x)dx$$

tenglikka ega bo'lamiz. Lobachevskiy formulasi isbotlandi.

Natiga. Agar $f(x)=1$ bo'lsa,

$$\int_0^\infty \frac{\sin^2 x}{x^2}dx = \int_0^\infty \frac{\sin x}{x}dx = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$$

tenglik o'rinli bo'ladi.

2-teorema. Agar $f(x+\pi) = f(x)$ va $f(\pi - x) = f(x)$, $0 \le x < \infty$ shartlarni qanoatlantirsa va ushbu

$$\int_0^\infty \frac{\sin^4 x}{x^4} f(x)dx$$

integral xosmas Riman integrallar fazosida aniqlangan bo'lsa, quyidagi tenglik o'rili bo'ladi.

$$\int_0^\infty \frac{\sin^4 x}{x^4} f(x)dx = \int_0^{\frac{\pi}{2}} f(t)dt - \frac{2}{3}\int_0^{\frac{\pi}{2}} \sin^2 t f(t)dt .$$

Isbot. Bizga ma'lumki,

$$\frac{d^2}{dx^2}\left(\frac{1}{\sin^2 x}\right) = \frac{6}{\sin^4 x} - \frac{4}{\sin^2 x}$$

tenglik o'rinli bo'ladi.

2-lemmandagi tenglikning o'ng tomonidagi 2 marta hadma-had differensiallab, biz quyidagi tenglikka ega bo'lamiz.

$$\frac{1}{\sin^4 a} - \frac{2}{3\sin^2 a} = \frac{1}{a^4} + \sum_{m=1}^{\infty}\left(\frac{1}{(a-m\pi)^4} + \frac{1}{(a+m\pi)^4}\right)$$

1-teoremani isbotlashda foyadalanilgan usulni qo'llab quyidagi tenglikni yozishimiz mumkin.

$$\int_0^\infty \frac{\sin^4 x}{x^4} f(x)dx = \int_0^{\frac{\pi}{2}} \sin^4 t\left(\frac{1}{\sin^4 t} - \frac{2}{3\sin^2 t}\right) f(t)dt$$

tenglikka ega bo'lamiz. Bundan esa biz isbotlayotgan

$$\int_0^\infty \frac{\sin^4 x}{x^4} f(x)dx = \int_0^{\frac{\pi}{2}} f(t)dt - \frac{2}{3}\int_0^{\frac{\pi}{2}} \sin^2 t f(t)dt$$

tenglik kelib chiqadi.

Natija. Agar $f(x)=1$ bo'lsa

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$$\int_0^{\infty} \frac{\sin^4 x}{x^4} dx = \frac{\pi}{3}$$

tenglik o‘rinli bo‘ladi.

Natiga. Agar $f(x)$ funksiya $f(x+\pi) = f(x)$ va $f(\pi - x) = f(x)$,$0 \le x < \infty$ shartlarni qanoatlantirsin.

$$I = \int_0^{\infty} \frac{\sin^{2n+1} x}{x} f(x) dx = \int_0^{\infty} \frac{\sin x}{x} \sin^{2n} x f(x) dx$$

Agar $g(x) = \sin^{2n} x f(x)$ desak , $g(x+\pi) = g(x)$ va $g(\pi - x) = g(x)$, $0 \le x < \infty$ shartlar o‘rinli bo‘ladi. Agar $f(x) = 1$ desak, 1-teoremaga ko‘ra

$$\int_0^{\infty} \frac{\sin^{2n+1} x}{x} dx = \int_0^{\frac{\pi}{2}} \sin^{2n} x dx = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2}$$

kelib chiqadi.

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