Termiz muxandislik texnologiya instituti akademik litseyi katta o'qituvchisi
Diyorova Gulnoza Baxtiyor qizi
@gulnozadiyorova897gmail.com

FILTRATSIYA TENGLAMASI UCHUN BITTA CHEGARAVIY MASALA

Parabolik tipdagi tenglamalarning turli ko'rinishlari uchun chegaraviy masala qo'yish juda ko'plab avtorlar tomonidan o'rganilgan. Hozirgi kunda zamonaviy fanning yutuqlari, shu bilan birgalikda ishlab chiqarishning turli masalalari hamda fizika, mexanika, texnika, biologiya, ekologiya va sotsiologiya kabi fanlarning juda ko'plab muammolarining matematik modellari parabolik tipdagi tenglamalarning turli ko'rinishlari uchun nolokal (sohaning chegaralarida funksiyaning qiymati berilmasdan, balki sohaning u yoki qismi orasidagi bog'lanishlar beriladi) masalalarni o'rganishni talab qilmoqda. Nolokal masalalar noklassik masalalar jumlasiga kirib, noklassik masalalar bilan hozirgi kunda dunyoning turli mamlakatlarida juda ko'plab ilmiy maktablar, universitetlar olimlari tomonidan ilmiy izlanishlar olib borilmoqda.

Masalaning qo'yilishi. Masala yechimining yagonaligi

Ushbu paragrafda parabolik tipdagi tenglamalarning eng sodda vakili bo'lgan filtratsiya tenglamasi uchun bitta chegaraviy masala o'rganiladi. Masala yechimining yagonaligi ekstremum prinsipidan foydalanib ko'rsatildi, yechimning mavjudligi esa issiqlik potensiallari usuli yordamida ko'rsatiladi.

1. Masalaning qo'yilishi. $D=\{(x,t):0<x<l,0<t<T\}$ ***sohada***

$$u_t(x,t)=u_{xx}(x,t)\,,\ (x,t)\in D \qquad \textbf{(1)}$$

tenglamaning

$$u(x,0)=\varphi(x)\,,\ \ 0\le x\le l \qquad \textbf{(2)}$$

boshlang'ich va chegaraviy

$$u(0,t)=\alpha u(x_0,t)\ \ ,\quad 0\le t\le T \qquad (3)$$

$$u_x(l,t)=\psi(x)\ , \quad 0\le t\le T \qquad (4)$$

shartlarni qanoatlantiruvchi yechimi topilsin.

Shu bilan birgalikda (1) - (4) masalada quyidagilarni berilgan funksiyalar deb qabul qilamiz:

1. $\varphi(t),\ \psi(t)$ - uzluksiz funksiyalar;

2. $\alpha\ ,\ x_0$ - musbat o'zgarmaslar bo'lib, quyidagi tengsizliklarni

qanoatlantirsin

$0<\alpha\le 1\ , \quad 0<x_0<1 \quad ;$

3. Quyidagi kelishuvlik shartlari bajarilsin: $\varphi(0)=\alpha\varphi(x_0)\ ,\ \varphi'(l)=\psi(0)$.

(4) - ko'rinishdagi ichki nolokal shartli masalalar populyatsiyaning ko'payish strukturasining yozilishidan kelib chiqqan [14]. Ichki nolokal shartli masalalar turli ko'rinishdagi parabolik tipdagi tenglamalar uchun juda ko'plab avtorlar tomonidan o'rganilgan hamda bunga doir umumiy ma'lumotlarni [6] dan olish mumkin.

2. Aprior baholar.

Lemma 1. Agar 1, 2, 3 - shartlar bajarilsa, u holda (1) - (4) masalaning yechimi uchun ushbu

$$|u(x,t)|\le M\ , \ (x,t)\in D \qquad \textbf{(*)}$$

baho o'rinli bo'ladi, bu erda $M=\max\left\{\max_{x\in[0,t]}|\varphi(x)|,\ \max_{t\in[0,t]}|\psi(x)|\right\}$ ga teng.

Isbot. Lemma 1 ning isboti bevosita (1)-(4) masalaning shartidan 1.2.3. shartlardan va dissertatsiyaning 1-bobi 2-paragrafidagi parabolik tipdagi tenglamalar uchun keltirilgan ekstremum prinsipi va uning 2-xossasidan ya'ni agar $u(x,t)$ funksiya issiqlik tarqalish tenglamasining yechimi bo'lsa, u holda $\forall(x,t)\in Q$ *uchun* quyidagi munosabat $|u(x,t)|\le\max_{\Gamma}|u(x,t)|$ o'rinli ekanligidan $(x,t)\in D$ sohada (1) - (4) masala yechimi uchun

$$|u(x,t)|\le\max\left\{\max_{x\in[0,t]}|\varphi(x)|\max_{t\in[0,t]}|\varphi(x)|\right\}=M$$

baho o'rinli bo'ladi.

Lemma 1. isbot bo'ldi.

3. Masala yechimining yagonaligi.

Teorema 1. Agar (1) - (4) masalaning yechimi mavjud bo 'lib, (1)-(2) shartlar bajarilsa, u holda (1) - (4) masalaning echimi yagonadir.

Isbot. Faraz qilaylik, D sohada (1) - (4) masala ikkita yechimga bo'lsin, ya'ni $u_1(x,t)$ va $u_2(x,t)$ bo'lsin. U holda ularning ayirmasi ham yechim bo'ladi.

$$\upsilon(x,t)=u_1(x,t)-u_2(x,t)$$

Bundan $\upsilon(x,t)$ funksiya uchun D sohada quyidagi masalani olamiz:

$$\upsilon(x,t)=u_1(x,t)-u_2(x,t)\ ,\quad (x,t)\in D \qquad (5)$$

tenglamaning

$$\upsilon(x,0)=0\ ,\quad 0\le x\le l \qquad (6)$$

boshlang'ich va chegaraviy

$$\upsilon(0,t)=\alpha\upsilon(x_0,t)\ ,\quad 0\le t\le T \qquad (7)$$

$$\upsilon_x(l,t)=0\ ,\quad 0\le t\le T \qquad (8)$$

shartlarni qanoatlantiruvchi yechimi topilsin.

Endi yangi hosil bo'lgan (5)-(8) masalani yechib ko'ramiz. Bu masala yechimining aynan nolga teng ekanligini ko'rsatamiz. Buning uchun energiya integrali usulidan foydalanishimiz mumkin. (5) tenglamada quyidagi operatorni qaraylik

$$L(\upsilon)=\upsilon_t(x,t)-\upsilon_{xx}(x,t)=0. \qquad (9)$$

(9) operatorni $\upsilon(x,t)$ ga ko'paytirsak, quyidagi ko'rinishga kelib qoladi:

$$\upsilon L(\upsilon)=\upsilon\upsilon_t-\upsilon\upsilon_{xx}=0. \qquad (10)$$

(10) tenglikni $\overline{D}$ soha bo'yicha integrallaymiz.

$$\iint_{\overline{D}} \upsilon L(\upsilon)dxdt = \iint_{\overline{D}} (\upsilon\upsilon_t - \upsilon\upsilon_{xx})dxdt = 0 \qquad (11)$$

(11) tenglikni quyidagi divergent ko'rinishda ifodalab olamiz.

$$\upsilon\upsilon_t - \upsilon\upsilon_{xx} = \frac{\partial}{\partial t}\left(\frac{1}{2}\upsilon^2\right) - \frac{\partial}{\partial x}(\upsilon\upsilon_x) + \upsilon_x^2 \qquad (12)$$

(12) ifodani (11) ga olib borib qo'ysak, quyidagi tenglikka kelamiz:

$$\iint_{\overline{D}} \left(\frac{\partial}{\partial t}\left(\frac{1}{2}\upsilon^2\right) - \frac{\partial}{\partial x}(\upsilon\upsilon_x) + \upsilon_x^2\right)dxdt = 0$$

$$\iint_{\overline{D}} \left(\frac{\partial}{\partial t}\left(\frac{1}{2}\upsilon^2\right) - \frac{\partial}{\partial x}(\upsilon\upsilon_x)\right)dxdt + \iint_{\overline{D}} \upsilon_x^2 dxdt = 0$$

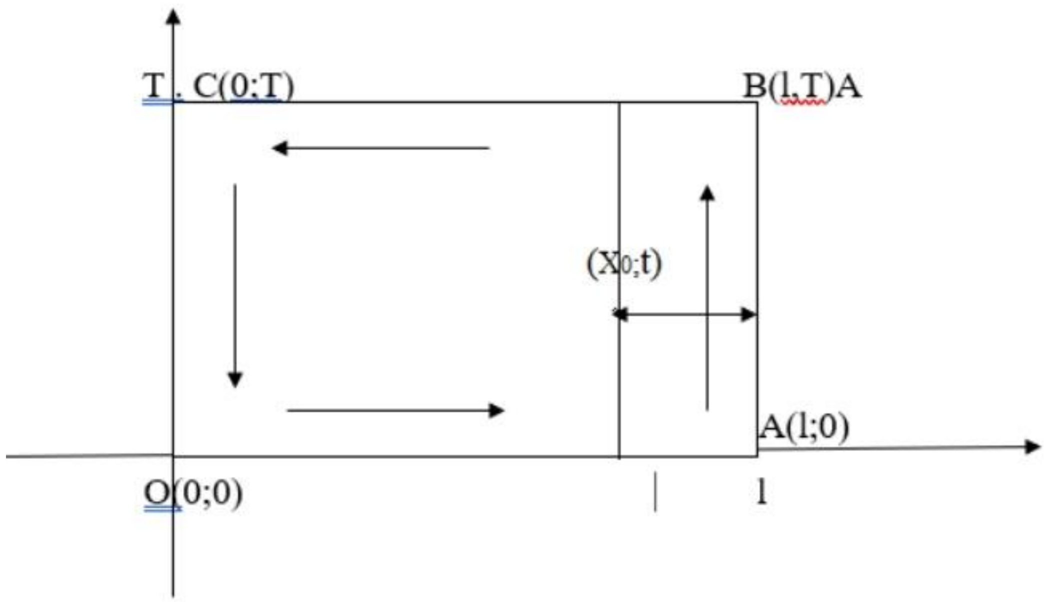
$$I_1 + I_2 = 0$$

$$I_1 = \iint_{\overline{D}} \left(\frac{\partial}{\partial t}\left(\frac{1}{2}\upsilon^2\right) - \frac{\partial}{\partial x}(\upsilon\upsilon_x)\right)dxdt \quad . \qquad (13)$$

(13) integralni hisoblash uchun Grin formulasidan foydalanamiz.

$$\iint_{Q} \left(\frac{\partial}{\partial x}P(x,y) - \frac{\partial}{\partial y}Q(x,y)\right)dxdy = \iint_{L} Qdx + Pdy$$

$$\iint_{\overline{D}} \left(\frac{\partial}{\partial t}\left(\frac{1}{2}\upsilon^2\right) - \frac{\partial}{\partial x}(\upsilon\upsilon_x)\right)dxdt = \iint_{\Gamma} \upsilon\upsilon_x dt + \frac{1}{2}\upsilon^2 dx = I_{OA} + I_{AB} + I_{CO} \qquad (14)$$



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(14) ning har bir integralini hisoblab chiqaramiz.

OA chiziqda $t=0$, $dt=0$, $0\le x\le l$, $\upsilon(x,0)=0$ lar o‘rinli.

$$I_{OA}=\int_0^l \frac{1}{2}\upsilon^2(x,0)dx=0$$

AB chiziqda $x=l$, $dx=0$, $0\le t\le T$, $\upsilon_x(l,t)=0$ tengliklar o‘rinli.

$$I_{AB}=\int_0^T \upsilon(l,t)\cdot\upsilon_x(l,t)dt=0$$

BC chiziqda esa $t=T$, $dt=0$, $l\le x\le 0$

$$I_{BC}=-\int_0^l \frac{1}{2}\upsilon^2(x,T)dx$$

CO chiziqda $x=0$, $dx=0$, $T\le t\le 0$, $\upsilon(0,t)=\alpha\upsilon(x_0,t)$ tengliklar o‘rinli.

$$I_{CO} = -\int_0^T \upsilon(0,t)\cdot\upsilon_x(0,t)dt = -\int_0^T \alpha\upsilon(x_0,t)\cdot\upsilon_x(0,t)dt$$

Yuqoridagi integrallarni (3.14) ga olib borib qo‘yamiz va quyidagi tenglik kelib chiqadi:

$$\iint_D \left(\frac{\partial}{\partial t}\left(\frac{1}{2}\upsilon^2\right) - \frac{\partial}{\partial x}(\upsilon\upsilon_x)\right)dxdt = \int_0^T \upsilon(l,t)\cdot\upsilon_x(l,t)dt - \int_0^t \frac{1}{2}\upsilon^2(x,T)dx = 0 \quad (15)$$

(15) tenglik va integralning xossalaridan $\upsilon(x,t)=0$ ekanligi kelib chiqadi.

Demak, $u_1(x,t)=u_2(x,t)$ ekan, ya’ni farazimiz noto‘g‘ri hamda tenglama yagona yechimga ega ekan.

Teorema 1. isbot bo‘ldi.

2. Masala yechimining mavjudligi

Magistirlik dissertatsiya ishining ushbu paragrafida filtratsiya tenglamasi uchun qo‘yilgan bitta chegaraviy masala yechimining mavjudligini Grin funksiyasi yordamida keltiramiz.

Issiqlik tarqalish tenglamasi uchun chegarasi Γ sirtdan iborat bo‘lgan biror D sohada birinchi chegaraviy masalaning Grin funksiyasi deb $M(\upsilon)=0$ tenglamani va

1) $\upsilon(\xi,\tau)\big|_{\tau=t} = 0$

2) $\upsilon(\xi,\tau)\big|_{PA} = \upsilon(\xi,\tau)\big|_{BQ} = G_0(x,t;\xi,\tau)$

shartlarni qanoatlantiruvchi yechimni topishga aytiladi.

Tenglama uchun Grin formulasidan foydalanib (1) – (4) masala yechimining integral ifodasini keltirib chiqaramiz.

Faraz qilaylik $u(x,t)$ funksiya (1) - (4) masalaning yechimi bo‘lsin va

$$G_0(x,t;\xi,\tau) = \frac{1}{2\sqrt{\pi(t-\tau)}}e^{-\frac{(x-\xi)^2}{4(t-\tau)}}$$

issiqlik tarqalish tenglamasining fundamental yechimi bo‘lsin.

(1) tenglamada x ni ξ ga t ni η ga almashtiramiz:

$$u_{\xi\xi} - u_\eta = 0\ ,\ u_{\xi\xi}(\xi,\eta) - u_\eta(\xi,\eta) = 0\ .$$

Hosil bo‘lgan funksiyani Grin funksiyasiga ko‘paytiramiz va unga qo‘shma bo‘lgan

$$M(G) = G_{\xi\xi} + G_\eta = 0$$

tenglamani esa $u(\xi,\eta)$ ga ko‘paytiramiz.

Hosil bo‘lgan tenglamalarni hadlab ayiramiz.

$$\begin{cases} G(x,t;\xi,\eta)\cdot u_{\xi\xi}(\xi,\eta) - G(x,t;\xi,\eta)\cdot u_\eta(\xi,\eta) = 0 \\ u(\xi,\eta)\cdot G(x,t;\xi,\eta) + u(\xi,\eta)\cdot G(x,t;\xi,\eta) = 0 \end{cases}$$

$$G(x,t;\xi,\eta)\cdot u_{\xi\xi}(\xi,\eta) - G(x,t;\xi,\eta)\cdot u_\eta(\xi,\eta) -$$
$$u(\xi,\eta)\cdot G_{\xi\xi}(x,t;\xi,\eta) - u(\xi,\eta)\cdot G_\eta(x,t;\xi,\eta) = 0$$

Bu tenglikni quyidagi ko‘rinishda yozib olamiz:

$$\frac{\partial}{\partial\xi}\big(G(x,t;\xi,\eta)\big)\cdot u_\xi(\xi,\eta) - u(\xi,\eta)\cdot G_\xi(x,t;\xi,\eta) - \frac{\partial}{\partial\eta}\big(G(x,t;\xi,\eta)\cdot u(\xi,\eta)\big) = 0 \quad (16)$$

(16) tenglik tenglamaning Divergent yechimi bo’ladi, ya’ni yechimning Divergent ko‘rinishi hisoblanadi.

$G_0(x,t;\xi,\tau)$ funksiya $L(G_0)$ tenglamani x, t o‘zgaruvchilari bo‘yicha va unga qo‘shma bo‘lgan $M(G_0)$ tenglamani ξ , η o‘zgaruvchilari bo‘yicha qanoatlantiradi.

$$L(G_0) = G_{0xx} - G_{0t} = 0$$

$$G_{0x} = \frac{1}{2\sqrt{\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} \left(-\frac{2(x-\xi)}{4(t-\tau)}\right) = -\frac{x-\xi}{4\sqrt{\pi}(t-\tau)^{\frac{3}{2}}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} \quad (17)$$

$$G_{0xx}=-\frac{1}{4\sqrt{\pi}\left(t-\tau\right)^{\frac{3}{2}}}e^{-\frac{(x-\xi)^2}{4(t-\tau)}}=+\frac{2\left(x-\xi\right)^2}{16\sqrt{\pi}\left(t-\tau\right)^{\frac{5}{2}}}e^{-\frac{(x-\xi)^2}{4(t-\tau)}} \qquad (18)$$

$$G_{0t}=-\frac{1}{4\sqrt{\pi}\left(t-\tau\right)^{\frac{3}{2}}}e^{-\frac{(x-\xi)^2}{4(t-\tau)}}=+\frac{2\left(x-\xi\right)^2}{16\sqrt{\pi}\left(t-\tau\right)^{\frac{5}{2}}}e^{-\frac{(x-\xi)^2}{4(t-\tau)}} \qquad (19)$$

ekanligini e'tiborga olsak, $L\left(G_0\right)=0$ bo'ladi. Xuddi shunday $M\left(G_0\right)=0$ ekanligini ko'rishimiz mumkin.

Faraz qilaylik, $M\left(x,t\right)$ va $M_1\left(x,t+\varepsilon\right)$ nuqtalar D sohaning ichidan olingan biror fiksirlangan nuqtalari bo'lsin. Bu yerda $h>0$ son $u\left(x,t\right)$ funksiyaning qiymatini $M\left(x,t\right)$ nuqtada aniqlaymiz. $G_0\left(x,t+\varepsilon;\xi,\tau\right)$ va $\frac{\partial G_0}{\partial \xi}$ funksiyalarning D sohada h parametr bo'yicha uzluksizligidan $h\to 0$ limitga o'tamiz. Unda

$$\lim_{h\to 0}\int_{BQ}\frac{1}{2\sqrt{\pi\varepsilon}}e^{-\frac{(x-\xi)^2}{4\varepsilon}}u\left(\xi,\tau\right)d\xi=u\left(x,t\right) \qquad (20)$$

hosil bo'ladi. Quyidagi belgilashlarni kiritamiz:

$$\frac{x-\xi}{2\sqrt{\varepsilon}}=z\ , \qquad \xi=x-2\sqrt{\varepsilon z}\ , \qquad d\xi=-2\sqrt{\varepsilon}dz\,.$$

Bundan (20) quyidagi ko'rinishga kelib qoladi:

$$\lim_{h\to 0}\int_{BQ}\frac{1}{2\sqrt{\pi\varepsilon}}e^{-\frac{(x-\xi)^2}{4\varepsilon}}u\left(\xi,\tau\right)d\xi=\lim_{h\to 0}\left[\int_0^{+\infty}\frac{e^{-x^2}}{2\sqrt{\pi\varepsilon}}u\left(x-2\sqrt{\varepsilon z},t\right)\left(-2\sqrt{\varepsilon}\right)dz\right]=$$

$$=\frac{1}{\sqrt{\pi}}\int_0^{+\infty}\lim_{h\to 0}u\left(x-2\sqrt{\varepsilon z},t\right)e^{-x^2}dz=\frac{1}{\sqrt{\pi}}\int_0^{+\infty}u\left(x,t\right)e^{-x^2}dz=$$

$$=\frac{u\left(x,t\right)}{\sqrt{\pi}}\int_0^{+\infty}e^{-x^2}dz=\frac{u\left(x,t\right)}{\sqrt{\pi}}\sqrt{\pi}=u\left(x,t\right) \qquad (21)$$

Agar $PABQ$ sohaning AP va BQ egri chiziqli chegarasini t o'qiga parallel bo'lgan $x=0$ va $x=l$ to'g'ri chiziqlarga almashtirsak, natijada $ABPQ$ to'g'ri to'rtburchakli soha hosil bo'ladi. Bu to'g'ri to'rtburchak sohada issiqlik tarqalish tenglamasi uchun birinchi chegaraviy masalaning ushbu ko'rinishdagi

$$u(x,t)=\int_0^l G(x,t;\xi,0)\varphi(\xi)d\xi+\int_0^t \frac{\partial G(x,t;\xi,0)}{\partial \xi}\Bigg|\mu_1(\tau)d\tau-$$

$$-\int_0^t \frac{\partial G(x,t;\xi,\tau)}{\partial \xi}\Bigg|_{\xi=l}\mu_2(\tau)d\tau \qquad (22)$$

yechimi hosil bo'ladi. Bu yerda $G(x,t;\xi,0)$ birinchi chegaraviy masala uchun Grin funksiyasi bo'lib, bu funksiya quyidagi ko'rinishda bo'ladi:

$$G(x,t;\xi,0)=\frac{1}{2\sqrt{\pi(t-\tau)}}\left[e^{-\frac{(x-\xi)^2}{4(t-\tau)}}-e^{-\frac{(x+\xi)^2}{4(t-\tau)}}\right]. \qquad (23)$$

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