Acumen: International Journal of Multidisciplinary Research
Volume 1, Issue 1

Some basic concepts and definitions of graph theory

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Abstract. The development of graph theory and networks has led to the formation of junctions between mathematics and economics. In two established different areas of research in this field: algebraic and optimization. Taking into account the requirements of logistics research applications, we will continue to adhere mainly to the second of these areas. For this purpose, we decided to devote this work to the study of the basic elements of graph theory. In it, we have given the basic concepts such as arc, loop, path, cycle, contour, simple chain and others.

Keywords. Arc, loop, path, cycle, contour, simple chain.

The development of graph and network theory has led to the formation of two distinct areas of research in this field: algebraic and optimization. Considering the requirements of logistics research applications, we will mainly adhere to the latter of these directions. Below, we will briefly present the basic concepts and definitions related to this subject area [1]-[3].

Let:

- V be a certain set $V=\{v_1,v_2,...,v_n\}$, whose elements v_i are called **vertices** (for convenience, we will sometimes denote vertices by numbers 1, 2, … or letters $a,b, ..., x, y$, etc.);
- U be the set of ordered pairs of vertices (v_i,v_j) (from the set V), whose elements are called **arcs**, where v_i is the start and v_j is the end of such an arc. Regarding the arc (v_i,v_j), it is also said that it "originates" from vertex v_i and "enters" vertex v_j. For convenience, we will sometimes denote arcs by Greek letters α,β,γ, etc.

The set of the aforementioned sets V and U is called a graph (of order n according to the number of vertices) and is denoted by $G(V,U)$. If a graph of order n has m arcs, such a graph is usually called an $(n;m)$-graph. In the schematic representation of a graph, its vertices are represented by points (often circles), and arcs are depicted as arrows connecting specific vertices (generally indicating the direction). The lengths of these arrows and their type (straight lines or not) do not matter.

The requirement of ordering pairs of vertices for the set of arcs U in the definition above corresponds to the concept of so-called **directed** graphs. For certain problems in graph theory and their corresponding logistical applications, it is not necessary to distinguish the start and end of an arc (i.e., it is not necessary to assign specific directions to the arcs). In this case, the graph is called **undirected**. In undirected graphs, the arcs are referred to as **edges**.

A loop is an arc for which the starting and ending vertices are the same.

Two vertices are called **adjacent** if there is an arc connecting them (its direction does not matter).

A vertex and an arc are said to be **incident** (to each other) if the arc "enters" or "originates" from this vertex.

Two arcs are called **incident** (to each other) if they are incident to the same vertex.

To represent the following concepts, we will use some "conditional" arbitrary sequence of vertices: $\{x_1, x_2, ..., x_n\}$, where any member of the sequence is an element of the set of vertices, i.e., $x_i \in V, i = 1, 2, ..., n+1$

A chain is any sequence of arcs of the form $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ such that the endpoints (i.e., the starting point and the ending point) of arc α_i are vertices x_i and x_{i+1} (in any order). In other words, for all $i = 1, 2, ..., n$ we have:

- either $\alpha_i = (x_i; x_{i+1})$;
- or $\alpha_i = (x_{i+1}; x_i)$.

Vertex x_1 is called **the initial vertex** of the chain, and vertex x_{n+1} is the terminal vertex of the corresponding chain. It is also said that the chain connects the initial vertex (x_1) with the terminal vertex (x_{n+1}).

The length of a chain is the number equal to the number of arcs it contains.

A path is a chain for which, for all $i = 1, 2, ..., n$, the arcs α_i have only the following form: $\alpha_i = (x_i; x_j)$, i.e., it is a chain that includes only **direct arcs** (the direction of such arcs coincides with the direction of the sequence of vertices).

A cycle is a chain for which the initial vertex (x_1) and the terminal vertex (x_{n+1}) are the same.

A contour is a path for which the initial vertex (x_1) and the terminal vertex (x_{n+1}) are the same (i.e., a contour is a cycle that is also a path).

A simple chain is a chain for which no vertex in the corresponding sequence $\{x_1, x_2, ..., x_n, x_{n+1}\}$ is incident to more than two arcs contained in this chain.

Similarly, the concepts **of a simple path**, **a simple cycle**, and **a simple contour** are introduced. The term "simple" emphasizes that the corresponding chain (path, cycle, contour) does not contain any cycles within itself.

Two vertices are called **connected** if there is a chain connecting them (note that the direction of arcs in such a chain does not matter for this definition).

A graph is called **connected** if any two of its vertices are connected.

The relation introduced above (a binary relation defined on pairs of vertices) of being connected is reflexive, symmetric, and transitive. Therefore, it is an equivalence relation. It partitions the entire set of vertices of the graph into disjoint subsets called the **connected components** of the graph.

Note that in the classical approach to the theory of directed graphs, the concept of connectivity is introduced differently. Let's illustrate this. First, the concept of a **related graph** (or the concept of a **base**) for the corresponding directed graph is introduced. This is the graph that results if the orientation is removed from the corresponding directed graph. For the related graph (i.e., for the base of the corresponding directed graph), concepts similar to those presented above are introduced, but with the prefix "semi-". For example, the concept of a **semi-chain**: this is a chain for the resulting related graph. Similarly, the concepts of a **semi-cycle**, a **semi-path**, etc., are introduced.

To define connectivity (in the classical theory of directed graphs), the concept of "reachability" is also introduced. A vertex v_j is said to be reachable from a vertex v_i if there is a simple path starting at v_i and ending at v_j (we denote this situation as $v_i \to v_j$). It is easy to see that such a relation ("reachability") in a directed graph is transitive but not symmetric and not reflexive. Therefore, it is not an equivalence relation. Consequently, in the theory of directed graphs, the following different concepts of connectivity are introduced: strong connectivity, one-way connectivity, weak connectivity, and disconnection.

A directed graph is called **strongly connected** if for any pair of its vertices v_i and v_j, both of the following conditions are met: 1) $v_i \to v_j$; 2) $v_j \to v_i$.

A directed graph is called **one-way connected** if for any pair of its vertices v_i and v_j, at least one of the following conditions is met: 1) $v_i \to v_j$; 2) $v_j \to v_i$.

A directed graph is called **weakly connected** if for any pair of its vertices v_i and v_j, there exists a semi-chain connecting them, i.e., the corresponding related graph is connected.

Examples of such graphs are shown in Figure 1 [4].

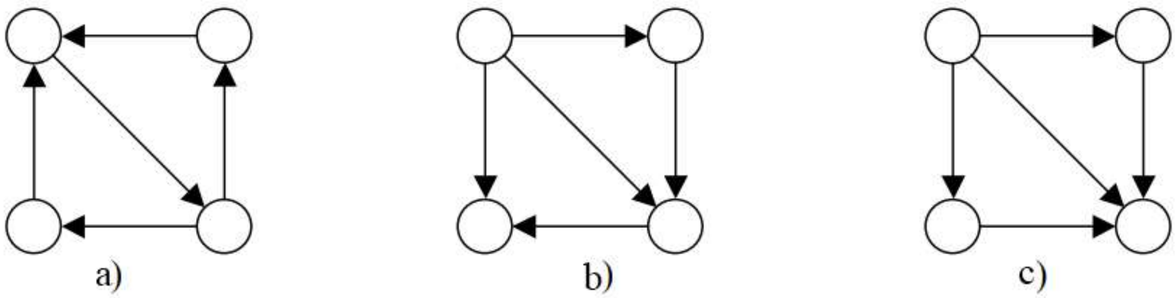

Figure 1 [2]. Illustration of types of connectivity for directed graphs: a) strongly connected b) one-way connected c) weakly connected.

To identify the specified types of connectivity in a directed graph, the following propositions may be useful.

PROPOSITION 1. A directed graph is strongly connected if and only if it contains a spanning contour (i.e., a cyclic path that passes through all vertices of the graph).

PROPOSITION 2. A directed graph is one-way connected if it contains a spanning path.

PROPOSITION 3. A directed graph is weakly connected if it contains a spanning semi-chain.

LITERATURE

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